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L-GROUPS AND THE LANGLANDS PROGRAM FOR COVERING GROUPS

L-groups and parameters for covering groups

Martin H. Weissman

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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L-GROUPS AND PARAMETERS FOR COVERING GROUPS

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Martin H. Weissman

Abstract. — We incorporate covers of quasisplit reductive groups into the Langlands program, defining an L-group associated to such a cover. We work with all covers that arise from extensions of quasisplit reductive groups by $\mathbf{K}_2$—the class studied by Brylinski and Deligne. We use this L-group to parameterize genuine irreducible representations in many contexts, including covers of split tori, unramified representations, and discrete series for double covers of semisimple groups over $\mathbb{R}$. An appendix surveys torsors and gerbes on the étale site, as they are used in the construction of the L-group.

Résumé **(L-groupes et paramètres pour les revêtements de groupes).** — Nous intégrons des revêtements de groupes réductifs quasi-déployés dans le programme de Langlands, en définissant un L-groupe associé à un tel revêtement. Nous travaillons avec tous les revêtements qui résultent d'extensions de groupes réductifs quasi-déployés par $\mathbf{K}_2$ — la classe étudiée par Brylinski et Deligne. Nous utilisons ce L-groupe pour paramétrer des représentations irréductibles spécifiques dans de nombreux contextes, incluant les revêtements de tores déployés, les représentations sphériques, et les séries discrètes pour les revêtements doubles de groupes semi-simples réels. Une appendice étudie les torseurs et gerbes sur le site étale, puisqu'ils sont utilisés dans la construction du L-groupe.

Introduction

Constructions and conjectures. — Let $\mathbf{G}$ be a quasisplit reductive group over a local or global field F. In [**18**], Brylinski and Deligne introduce objects called *central extensions of* $\mathbf{G}$ *by* $\mathbf{K}_2$, and they express hope that for "a global field this will prove useful in the study of 'metaplectic' automorphic forms". We pursue their vision in this paper, and elaborate below.

Let n be a positive integer and let μ_n denote the group of nth roots of unity in F. Assume that μ_n has order n. Let $\mathbf{G}'$ be a central extension of $\mathbf{G}$ by $\mathbf{K}_2$, in the sense

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of [18]. We call the pair $\tilde{\mathbf{G}} := (\mathbf{G}', n)$ a "degree n cover" of $\mathbf{G}$. Fix a separable closure $\bar{F}/F$ and write $\mathrm{Gal}_F = \mathrm{Gal}(\bar{F}/F)$. Fix an injective character $\varepsilon \colon \mu_n \hookrightarrow \mathbb{C}^\times$.

Associated to $\mathbf{G}'$, Brylinski and Deligne associate three invariants, which we call Q, $\mathscr{D}$, and f. The first is a Weyl- and Galois-invariant quadratic form on the cocharacter lattice of a maximal torus in $\mathbf{G}$. The second is a central extension of this cocharacter lattice by $\mathscr{G}_m$ (in the category of sheaves of groups on $F_{\text{ét}}$). The third is difficult to describe here, but it reflects the rigidity of central extensions of simply-connected semisimple groups.

Part 1 of this article defines the L-group ${}^{\mathsf{L}}\tilde{G}$ of such a cover $\tilde{\mathbf{G}}$, from the three invariants Q, $\mathscr{D}$, and f (as well as n, ε, and $\bar{F}$). It is an extension of Gal_F by $\tilde{G}^\vee$, where $\tilde{G}^\vee$ is a pinned complex reductive group, on which Gal_F acts by pinned automorphisms. Unlike Langlands' L-group, ours does not come equipped with a distinguished splitting—although a noncanonical splitting often exists. Throughout the construction of ${}^{\mathsf{L}}\tilde{G}$, the arithmetically-inclined reader may replace $\mathbb{C}$ by any $\mathbb{Z}[1/n]$-algebra Ω endowed with $\varepsilon \colon \mu_n \hookrightarrow \Omega^\times$, without running into much difficulty. The dual group $\tilde{G}^\vee$ has been considered by other authors, and it appears in various forms in [**26**], [**51**], [**60**], and [**1**]. We tabulate the dual groups; each comes equipped with a 2-torsion element $\tau_Q(-1)$ in its center $\tilde{Z}^\vee$.

In our previous article [**78**], we limited our attention to split reductive groups, and constructed an L-group by bludgening Hopf algebras with two "twists". The construction here is more delicate, and more general. The "first twist" of [**78**] is encoded here in the following way. The quadratic Hilbert symbol may be used to define a canonical 2-cocycle, yielding an extension $\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_F \twoheadrightarrow \mathrm{Gal}_F$ which we call the metaGalois group. The metaGalois group may be of independent interest—one might look for its representations in nature, e.g., in the étale cohomology of a variety over $\mathbb{Q}(i)$ equipped with twisted descent data to $\mathbb{Q}$. Pushing out the metaGalois group via the central 2-torsion element in $\tilde{G}^\vee$ yields the first twist,

$$\tilde{Z}^\vee \hookrightarrow (\tau_Q)_* \widetilde{\mathrm{Gal}}_F \twoheadrightarrow \mathrm{Gal}_F \,. \tag{0.1}$$

The "second twist" of [**78**] provided the greatest challenge there and here. There, it was defined by twisting the multiplication in a Hopf algebra. After attempting many reformulations (e.g., a Tannakian approach), we found the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ on $F_{\text{ét}}$, which applies to covers of quasisplit groups and serves as the second twist here. It is a bit different from gerbes that typically arise in the Langlands program, and so we include an appendix with relevant background on torsors and gerbes. The étale fundamental group of this gerbe provides the second twist,

$$\tilde{Z}^\vee \hookrightarrow \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})) \twoheadrightarrow \mathrm{Gal}_F \,. \tag{0.2}$$

The Baer sum of (0.1) and (0.2) gives an extension of Gal_F by $\tilde{Z}^\vee$. Pushing out to $\tilde{G}^\vee$ (respecting the Gal_F-action throughout) yields the L-group

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_F,$$

of Part 1. In addition to its construction, we verify that this L-group behaves well with respect to Levi subgroups, passage between global fields, local fields, and rings of integers therein, and that the L-group construction is functorial for a class of "well-aligned" homomorphisms.

The construction of the L-group allows us to consider Weil parameters. When F is a local or global field, we consider the set $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/F)$ of $\tilde{G}^\vee$-orbits of Weil parameters $\mathcal{W}_F \to {}^{\mathsf{L}}\tilde{G}$. When $\mathcal{O}$ is the ring of integers in a nonarchimedean local field, we consider the set $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O})$ of $\tilde{G}^\vee$-orbits of unramified Weil parameters $\mathcal{W}_F \to {}^{\mathsf{L}}\tilde{G}$. One could similarly define Weil-Deligne parameters, (conjectural) global Langlands parameters, etc.

In a set of unpublished notes, we constructed an L-group for split groups without using the gerbe discussed above. This "$E_1 + E_2$" construction has been studied further by Wee Teck Gan and Gao Fan in [**29**] and [**30**]. The construction of this paper agrees with the $E_1 + E_2$ construction for split reductive groups; this is proven in a short note at the end of this volume.

With the construction of the L-group complete, we turn our attention to representation theory in Part 2. The cover $\tilde{\mathbf{G}}$ and character ε allow us to define ε-genuine irreducible representations of various sorts. The set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\bullet)$ is defined in three contexts.

- F **a local field:** Brylinski and Deligne construct [**18**, §10.3] a central extension $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G = \mathbf{G}(F)$, and we consider the set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F)$ of equivalence classes of irreducible admissible ε-genuine representations of $\tilde{G}$.
- F **a global field:** Brylinski and Deligne construct [**18**, §10.4] a central extension $\mu_n \hookrightarrow \tilde{G}_{\mathbb{A}} \twoheadrightarrow G_{\mathbb{A}} = \mathbf{G}(\mathbb{A})$, canonically split over $\mathbf{G}(F)$, and we consider the set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F)$ of equivalence classes of ε-genuine automorphic representations of $\tilde{G}_{\mathbb{A}}$.
- $\mathcal{O}$ **the integers in a nonarchimedean local field** F**:** Brylinski and Deligne construct [**18**, §10.3, 10.7] a central extension $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G = \mathbf{G}(F)$, canonically split over $G^\circ = \mathbf{G}(\mathcal{O})$. We consider the set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O})$ of equivalence classes of irreducible G°-spherical representations of $\tilde{G}$.

Part 2 introduces these classes of representations, reviewing or adapting foundational results as needed. These include old results, such as the basic theory of admissible, unitary, and tempered representations, and results which are recent for covering groups, such as the Satake isomorphism and Langlands classification. Some new features arise for covering groups: we introduce the notion of "central core character" which is a bit coarser than "central character". We place irreducible representations into "pouches" which should be subsets of L-packets in what follows later.

The remainder of the paper is devoted to supporting the following "Local Langlands Conjecture for Covers" (LLCC), an analog of the local Langlands conjectures (LLC). For the LLC, we refer to the excellent survey by Cogdell [**21**].

***Conjecture 0.1* (LLCC).** — When F is a local field, there is a *natural* finite-to-one parameterization,

$$\mathcal{L}_\varepsilon \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/F).$$

Such a conjecture is nearly meaningless, without defining the adjective "natural". Naturality in the (traditional) local Langlands conjectures (LLC) includes compatibility with the bijective parameterizations for split tori (class field theory) and unramified representations (the Satake isomorphism), and with parabolic induction (the Langlands classification). In [**11**, §10], it is suggested that naturality includes desiderata which specify how central characters and twisting by characters should correspond to various properties of and operations on Weil parameters. One could add to these desiderata today, specifying for example how the formal degree (of discrete series) should correspond to adjoint γ-factors for Weil parameters (see [**37**]), or how the contragredient operation should correspond to the Chevalley involution (see [**3**]).

For covers, we can make a similar list of desiderata for the LLCC. We expect a finite-to-one parameterization for covering groups, compatible with our results for split tori (described in Part 3), with the unramified case (described in Part 4), with parabolic induction (via the Langlands classification for covers) and central core characters and twisting by characters (described in Part 2), and with formal degrees and adjoint γ-factors.

Unlike the LLC, we have not attempted to characterize the image of our conjectural parameterization in this paper, i.e., we have not identified the "relevant" parameters for covering groups. The cases of split tori and discrete series for real groups should suggest a characterization in the future.

Part 3 focuses on the case of "sharp" covers of split tori. For such a sharp cover (over a local or global field, or in the unramified setting), we define a *bijective* parameterization

$$\mathcal{L}_\varepsilon \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/\bullet) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/\bullet).$$

This parameterization is natural for pullbacks of covers via isomorphisms, for isomorphisms of covers of a given split torus, and for Baer sums of covers. This case constrains and guides many others, and it occupies the largest part of this article. The "sharp" case quickly leads to the general case of split tori, where the parameterization may no longer be surjective.

Part 4 includes three more cases where a precise parameterization is possible. First is the spherical/unramified case. When $\tilde{\mathbf{G}}$ is a cover of a quasisplit group over $\mathcal{O}$, we define a *bijective* parameterization

$$\mathcal{L}_\varepsilon \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O}) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O}).$$

This parameterization follows from the Satake isomorphism (for covering groups), the parameterization for sharp covers of split tori above, and careful tracking of the Weyl group actions.