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ORDINARY PARTS OF ADMISSIBLE REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS II. DERIVED FUNCTORS

by

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Abstract. — If G is a connected reductive p-adic group and P is a parabolic subgroup of G, then we extend the functor Ord_P of ordinary parts to a certain δ-functor, which we denote $H^\bullet\mathrm{Ord}_P$. Using this functor, we compute certain Ext spaces in the category of admissible smooth representations of the group $\mathrm{GL}_2(\mathbf{Q}_p)$ over a finite field of characteristic p.

Résumé **(Parties ordinaires de représentations admissibles de groupes réductifs p-adiques II. Foncteurs dérivés)**

Soit G un groupe p-adique, connexe, réductif et P un sous-groupe parabolique de G ; nous étendons le foncteur Ord_P des parties ordinaires en un certain δ-foncteur, que nous notons $H^\bullet\mathrm{Ord}_P$. En utilisant ce foncteur, nous calculons certains Ext-espaces dans la catégorie des représentations lisses admissibles du groupe $\mathrm{GL}_2(\mathbf{Q}_p)$ sur un corps fini de caractéristique p.

1. Introduction

The goal of this paper, which is a sequel to [**10**], is to extend the functors of ordinary parts introduced in that paper to certain δ-functors. On the one hand, the δ-functors that we construct are easy to compute with in many situations; on the other hand, we expect that they coincide with the derived functors of the functors of ordinary parts. Since the functors of ordinary parts are right adjoint to the functors given by parabolic induction, we expect that the δ-functors introduced here will be useful in studying homological questions involving parabolically induced representations. As some evidence for this, we close the paper by applying them to make some Ext calculations in the category of admissible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over a finite field of characteristic p. These calculations play a role in the mod p and p-adic Langlands programme. (See [**6**] and [**9**].)

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To describe our results more precisely, let p be a prime, and A a finite Artinian local $\mathbb{Z}_p$-algebra of residue characteristic p. Let G be a connected reductive p-adic group, and P a parabolic subgroup of G, admitting a Levi factorization $P = MN$. We define a δ-functor $H^\bullet\mathrm{Ord}_P$ on the category of admissible smooth G-representations over A, taking values in the category of admissible smooth M-representations over A, such that $H^0\mathrm{Ord}_P \xrightarrow{\sim} \mathrm{Ord}_P$ (the functor of ordinary parts defined in [**10**, Def. 3.1.9]). The functors $H^\bullet\mathrm{Ord}_P$ are defined in terms of continuous N_0-cohomology, where N_0 is a compact open subgroup of N. This continuous cohomology is quite accessible to computation, and hence so are the functors $H^\bullet\mathrm{Ord}_P$. For example, one finds that the functors $H^i\mathrm{Ord}_P$ vanish for $i > \dim N$, and that $H^{\dim N}\mathrm{Ord}_P$ coincides (up to a twist) with the Jacquet functor of N-coinvariants. We conjecture that $H^\bullet\mathrm{Ord}_P$ is in fact a universal δ-functor, and thus coincides with the derived functors of Ord_P. In an appendix to this paper [**11**], joint with V. Paškūnas, we verify this conjecture in the case when $G = \mathrm{GL}_2(F)$, for any finite extension F of $\mathbb{Q}_p$.

In the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, P is a Borel subgroup of G, and $A = k$ is a field of characteristic p, we explicitly evaluate the functors $H^\bullet\mathrm{Ord}_P$ on all absolutely irreducible admissible smooth G-representations over k. As already mentioned, we are then able to apply the results of this calculation to compute certain Ext spaces in the category of admissible smooth G-representations over k. Some of the Ext computations that we make have also been made by Breuil and Paškūnas [**4**] and by Colmez [**6**], in both cases using different methods.

1.1. Arrangement of the paper. — Section 2 of the paper is devoted to recalling some results about injective objects in certain categories of representations, as well as describing the relationship between certain Ext functors and continuous cohomology. In Section 3 we define the δ-functors $H^\bullet\mathrm{Ord}_P$, and develop some of their basic properties. In Section 4 we study the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and P is a Borel subgroup in detail. Some of the more technical aspects of the calculations, which are not directly related to the general ideas of the paper, have been placed in an appendix. A second, separate, appendix, written jointly with V. Paškūnas, establishes an important homological result which is applied in Section 4.

1.2. Notation and terminology. — Throughout the paper, we fix a prime p. We let $\mathrm{Art}(\mathbb{Z}_p)$ denote the category of Artinian local $\mathbb{Z}_p$-algebras having finite residue field. All our rings of coefficients will be objects of $\mathrm{Art}(\mathbb{Z}_p)$.

We freely employ the terminology and notation that was introduced in the paper [**10**]. In particular, if V is a representation of a p-adic analytic group G over an object A of $\mathrm{Art}(\mathbb{Z}_p)$, then we let V_{sm} (resp. $V_{\mathrm{l.adm}}$) denote the G-subrepresentation of V consisting of the smooth (resp. locally admissible) vectors in V (see [**10**, Defs. 2.2.1, 2.2.15]).

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2. Injective and acyclic objects

Throughout this section, we let G denote a p-adic analytic group, and A an object of $\mathrm{Art}(\mathbb{Z}_p)$.

2.1. Injective objects.

— In this subsection we present some basic results regarding the existence and properties of injective objects in various categories of G-representations over A. These properties are largely standard, and presumably well-known. We recall them here, with proofs, for the sake of completeness.

2.1.1. *Proposition*. — *Each of the full subcategories* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ *and* $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ *of* $\mathrm{Mod}_G(A)$ *has enough injectives.*

Proof. — Certainly the category $\mathrm{Mod}_G(A)$ has enough injectives (since it coincides with the category of left modules over the ring $A[G]$). The functor $V \mapsto V_{\mathrm{sm}}$ is right adjoint to the natural embedding $\mathrm{Mod}_G^{\mathrm{sm}}(A) \hookrightarrow \mathrm{Mod}_G(A)$. Thus it takes injective objects in $\mathrm{Mod}_G(A)$ to injective objects in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. In particular, if V is an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and $V \hookrightarrow I$ an $A[G]$-linear embedding of V into an injective object of $\mathrm{Mod}_G(A)$, then the induced embedding $V \hookrightarrow I_{\mathrm{sm}}$ is an embedding of V into an injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. The proof in the case of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ proceeds similarly, using the functor $V_{\mathrm{l.adm}}$. □

2.1.2. *Proposition*. — *If I is an injective object of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$*, and H is an open subgroup of G, then I is also injective when regarded as an object of* $\mathrm{Mod}_H^{\mathrm{sm}}(A)$*.*

Proof. — The forgetful functor $\mathrm{Mod}_G^{\mathrm{sm}}(A) \to \mathrm{Mod}_H^{\mathrm{sm}}(A)$ is right adjoint to the exact functor $V \mapsto A[G] \otimes_{A[H]} V$. (This functor is naturally isomorphic to compact induction, $V \mapsto c\text{--}\mathrm{Ind}_H^G V$, and so takes smooth H-representations to smooth G-representations.) Thus it takes injectives to injectives. □

2.1.3. *Proposition*. — *If G is compact, then an inductive limit of injective objects of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ *is again an injective object of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$*.*

Proof. — Let $\{I_i\}$ be a directed system of injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, let $V \hookrightarrow W$ be an embedding of objects of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and suppose given a G-equivariant map

$$V \to \varinjlim_i I_i. \tag{2.1.4}$$

We must show that it extends to a map $W \to \varinjlim_i I_i$. By the usual argument with Zorn's lemma, it suffices to show that if $w \in W$, and if we write $W_0 = A[G]w$, then (2.1.4) extends to the G-subrepresentation $V + W_0$ of W. It is evidently equivalent to show that the restriction to $V_0 := V \bigcap W_0$ of (2.1.4) extends to W_0. Since G is compact and W is smooth, we see that W_0, and thus V_0, is finitely generated over A, and in particular over the completed group ring $A[[G]]$. Since $A[[G]]$ is Noetherian [**13**], V_0 is in fact finitely presented over $A[[G]]$. Thus the restriction to V_0 of (2.1.4) lifts to a map $V_0 \to I_i$ for some (sufficiently large) value of i. (See the following lemma.) We may then extend this map to W_0, since I_i is injective. Composing such an extension with the natural map $I_i \to \varinjlim_i I_i$, we obtain a G-equivariant map $W_0 \to \varinjlim_i I_i$ which extends the restriction to V_0 of (2.1.4), as required. □

The following lemma is a standard piece of algebra, whose proof we recall for the benefit of the reader.

2.1.5. *Lemma.* — *Let R be an associate ring with unit, and let M be a finitely presented left R-module. If $\{N_i\}$ is any directed system of left R-modules, then the natural map*

$$\varinjlim_i \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \varinjlim_i N_i) \tag{2.1.6}$$

is an isomorphism.

Proof. — Fix a finite presentation $R^s \to R^r \to M \to 0$ of M. Applying $\mathrm{Hom}_R(-, N_i)$ to this exact sequence, and taking into account the natural isomorphism

$$\mathrm{Hom}_R(R, N) \xrightarrow{\sim} N, \tag{2.1.7}$$

which holds for any R-module N, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, N_i) \to N_i^r \to N_i^s.$$

Passing to the inductive limit over all N_i yields an exact sequence

$$0 \to \varinjlim_i \mathrm{Hom}_R(M, N_i) \to \varinjlim_i N_i^r \to \varinjlim_i N_i^s.$$

Applying $\mathrm{Hom}_R(-, \varinjlim_i N_i)$ to the presentation of M, and again taking into account the natural isomorphism (2.1.7), we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, \varinjlim_i N_i) \to (\varinjlim_i N_i)^r \to (\varinjlim_i N_i)^s.$$

The preceding two exact sequences fit into the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varinjlim_i \operatorname{Hom}_R(M, N_i) & \longrightarrow & \varinjlim_i N_i^r & \longrightarrow & \varinjlim_i N_i^s \\ & & \downarrow{\scriptstyle (2.1.6)} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Hom}_R(M, \varinjlim_i N_i) & \longrightarrow & (\varinjlim_i N_i)^r & \longrightarrow & (\varinjlim_i N_i)^s, \end{array}$$

in which the second and third arrows are clearly isomorphisms. Thus so is the first, as claimed. □

2.1.8. *Remark*. — If the transition maps in the directed system $\{N_i\}$ are injective, then the map (2.1.6) will be an isomorphism provided that M is finitely generated. One can easily strengthen Lemma 2.1.5 to show that (2.1.6) is an isomorphism for every directed system $\{N_i\}$ if and only if M is a finitely presented R-module.

2.1.9. *Proposition*. — *If G is compact, then the category $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ has enough injectives, and injective objects in this category are also injective in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$.*

Proof. — If V is an admissible smooth G-representation, then its Pontrjagin dual V^* is a finitely generated $A[[G]]$-module. Fixing a surjection $A[[G]]^r \to V^*$, and dualizing, we obtain an embedding $V \hookrightarrow \mathscr{C}^{\mathrm{sm}}(G, A^*)^r$, where A^* is the Pontrjagin dual of A, thought of as an A-module, and $\mathscr{C}^{\mathrm{sm}}(G, A^*)$ denotes the space of smooth A^*-valued functions on G (regarded as a smooth admissible G-representation over A via the right regular G-action). Since $A[[G]]^r$ is free of finite rank as an $A[[G]]$-module, one easily verifies that $\mathscr{C}^{\mathrm{sm}}(G, A^*)$ is injective in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and so also in the category $\mathrm{Mod}_G^{\mathrm{adm}}(A)$. Thus any object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ embeds into an injective object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ that is also injective in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. In particular, $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ has enough injectives. The following lemma then implies that *every* injective object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ is injective in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. □

2.1.10. *Lemma*. — *Let $\mathscr{C}$ and $\mathscr{C}'$ be abelian categories, and let $F : \mathscr{C} \to \mathscr{C}'$ be an additive functor. Suppose that every object X of $\mathscr{C}$ admits a monomorphism $X \hookrightarrow J$ for some J such that $F(J)$ is an injective object of $\mathscr{C}'$. Then if I is any injective object of $\mathscr{C}$, the object $F(I)$ is injective in $\mathscr{C}'$.*

Proof. — Let I be an injective object of $\mathscr{C}$, and choose a monomorphism $I \hookrightarrow J$, where $F(J)$ is injective in $\mathscr{C}'$. The injectivity of I in $\mathscr{C}$ implies that this monomorphism splits, and thus that I is a direct summand of J. Consequently, $F(I)$ is a direct summand of $F(J)$, and hence is also injective in $\mathscr{C}'$. □

2.1.11. *Proposition*. — *If H is a compact subgroup of G, then any injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ is also injective as an object of $\mathrm{Mod}_H^{\mathrm{sm}}(A)$.*

Proof. — Since H is compact, we may find a compact open subgroup G' of G containing H. Appealing to Proposition 2.1.2, we see that we may replace G by G', and thus assume that G is also compact. We do so for the remainder of the proof.