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ON THE LIMIT OF FAMILIES OF ALGEBRAIC SUBVARIETIES WITH UNBOUNDED VOLUME

by

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Dedicated to José Manuel Aroca on the occasion of his 60^{th} birthday

Abstract. — We prove that the limit of a sequence of generic semi-algebraic sets given by a finite number of formulas always exists and is a semi-algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.

Résumé **(Sur la limite des familles de subvarietés algébriques sans volume borné)**
On prouve que la limite d'une suite d'ensembles semi-algébriques génériques donnés par un nombre fini de formules existe toujours et est un ensemble semi-algébrique, ensemble qui peut être donné explicitement comme une expression booléenne impliquant les primitives des formes additives de formules.

1. Introduction

Bishop [**2**] proved that the limit set of a sequence of complex purely k-dimensional algebraic subvarieties whose real volumes are uniformly bounded is again a purely k-dimensional algebraic subvariety. On the other hand, there are many reasons why one should be interested in analyzing the limit sets of algebraic subvarieties with unbounded volume. One reason is the existence of families of algebraic curves of increasing degree that are integrals of families of polynomials differential equations on the plane with bounded degree, a badly understood phenomenon related to the sixteenth Hilbert Problem (see [**4**], for instance). Another reason is that, despite the existence of topologically complicated limit sets of curves with unbounded volume (see [**6**], for instance), much can be said about the limit sets of algebraic subvarieties which lie in a family of subvarieties with finite complexity (see [**5**] for a definition of this concept).

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In this paper we consider the limit sets of one-parameter families of algebraic subvarieties, indexed by a natural number n, defined by a finite number of equations, each equation defined by a *formula*. Informally, a formula is a polynomial expression in which n appears in exponents only. Associated to each formula there is a *height*, which is the maximum number of nested n-th powers that appear in it. Here is the formal definition:

Definition 1. — *Formulas* and their *heights* are defined recursively as follows:

1. Every polynomial $F \in \mathbb{C}[X_1, \dots, X_m]$ is a formula of height zero.
2. If F_1 and F_2 are formulas, then $F_1 + F_2$ and F_1F_2 are formulas of height $\max(h_1, h_2)$, where h_i is the height of F_i.
3. If F is a formula of height h, then F^n is a formula of height $h + 1$.

A formula of height zero is also called a *primitive* formula; it is simply a complex polynomial.

At times we shall need to evaluate a formula F at a point $z \in \mathbb{C}^m$ and for a particular n. In this case, we shall write $F(z; n)$.

The height is a measure of the complexity of the formula: it measures how the degree increases with n. A formula of height h has degree proportional to n^h. More precisely, the degree of a formula of height h is $\Theta(n^h)$, using Landau's asymptotic notation as modified by Knuth [**3**].

An example of a formula of height 3 is

$$xy^2(((x^2 - y + 1)^n - 1)^n + x)^n + (xy)^n + (y^n - 1)^2 + 1.$$

Note that the degree is $2n^3 + 3 = \Theta(n^3)$.

The same polynomial family may be given by different formulas. For instance,

$$(x^n + y)^2 = (x^n)^2 + 2x^n y + y^2.$$

For our purposes, a convenient way to handle this issue is to express formulas in additive form. A formula is in *additive form* when it can be expressed as

$$Q_1A_1^n + Q_2A_2^n + \cdots + Q_lA_l^n - P,$$

where $Q_1, \dots, Q_l$, and P are primitive formulas and $A_1, \dots, A_l$ are arbitrary subformulas (necessarily of smaller height than the original formula). As we shall see later, additive forms help us to use induction on the height when working with formulas.

Lemma 1. — *Every formula can be written in additive form.*

Proof. — The proof is by induction on the number of operations required to obtain the formula according to Definition 1. If F is a primitive formula, then we can take $l = 0$ and $P = -F$. If $F = A^n$, then F is already in additive form because we can take $l = 1$, $Q_1 = 1$, $A_1 = A$, and $P = 0$. If $F = A + B$, then by induction A and B can be expressed in additive form, whose combination gives an additive form for F. If $F = AB$, then again by induction A and B can be expressed in additive form. By performing the multiplication AB on their additive forms, we get an additive form for F. □

As an example of the procedure described in the proof above, $(x^n+y)^2$ can be written in additive form as $(x^2)^n + (2y)x^n + y^2$. Note that the expression $(x^n)^2 + 2x^n y + y^2$ given earlier for $(x^n+y)^2$ is *not* in additive form.

Definition 2. — The *limit* (as $n \to \infty$) of a sequence (Ω_n) of subsets of $\mathbb{C}^m$ is the set $\lim \Omega_n$ of points that are limits of sequences of points lying in a subsequence of (Ω_n). More precisely,

$$\lim \Omega_n = \{\, z \in \mathbb{C}^m : \exists (z_n), z_n \to z, \exists (k_n), k_n \to \infty,\ z_n \in \Omega_{k_n} \text{ for sufficiently large } n \,\}.$$

Thus, according to this definition, the family of real curves $x^{2n} + y^{2n} = 1$ converges to the border of the unit square given by $x^2 \le 1$, $y^2 \le 1$. Actually, the definition of limit applies to the curves $x^n + y^n = 1$ (note that we now allow both even and odd exponents). These curves converge to the union of the border of the unit square with the two rays given by $x = -y$, $x^2 \ge 1$ (the curves actually alternate between these two limit sets, but our definition of limit covers this). Considered as a family of complex curves, $x^n + y^n = 1$ has as limit set the subset of $\mathbb{C}^2$ given by $\partial([|x| < 1] \cap [|y| < 1]) \cup [|x| = |y| > 1]$, as it is easy to verify.

We shall consider two situations: limit sets in $\mathbb{R}^m$ of families of algebraic subvarieties given by a finite number of formulas and limit sets in $\mathbb{C}^m$ of families of complex algebraic subvarieties.

In the real case it turns out that it is easier to describe the limits of semi-algebraic subsets, instead of algebraic subsets. Semi-algebraic subsets will also play a role in the complex case. An *algebraic subvariety* is the set of points that satisfy a polynomial equation $f(z) = 0$. For simplicity, we shall write this set as $[f = 0]$. A *semi-algebraic set* in $\mathbb{R}^m$ is one given by a Boolean expression on subsets of the form $[f > 0]$ or $[f \ge 0]$. We shall also deal with *basic closed semi-algebraic subsets*, which are the solutions of a system of polynomial inequalities: $[f_1 \ge 0, \ldots, f_k \ge 0]$, and with *basic open semi-algebraic subsets*, which are given by strict inequalities: $[f_1 > 0, \ldots, f_k > 0]$.

One main difficulty in the theory of semi-algebraic sets is that the closure of a basic open semi-algebraic set is not necessarily the corresponding basic closed semi-algebraic set obtained by relaxing the strict inequalities. That is, the closure of $[f_1 > 0, \ldots, f_k > 0]$ is not always $[f_1 \ge 0, \ldots, f_k \ge 0]$. Nor is the interior of a closed semi-algebraic set equal to the corresponding basic open semi-algebraic set obtained by restricting the inequalities. That is, the interior of $[f_1 \ge 0, \ldots, f_k \ge 0]$ is not always $[f_1 > 0, \ldots, f_k > 0]$. However, these statements are true *generically*, in two senses: (i) they are true if we perturb the polynomials slightly, and (ii) relaxing or restricting the inequalities only adds or removes lower dimensional components. We say that a basic closed semi-algebraic set is *generic* when it coincides with the closure of the corresponding basic open semi-algebraic set obtained by restricting the inequalities. In other words, a basic closed semi-algebraic set given by $[f_1 \ge 0, \ldots, f_k \ge 0]$ is generic when $[f_1 \ge 0, \ldots, f_k \ge 0] = \text{closure}[f_1 > 0, \ldots, f_k > 0]$. A *generic algebraic set* is, by definition, the boundary of a generic semi-algebraic subset. For a full discussion of real algebraic and semi-algebraic sets, see the book by Benedetti and Risler [**1**].

Our main result is the following:

Theorem 1. — *The limit of a sequence of generic semi-algebraic sets given by a finite number of formulas always exists and is a semi-algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.*

The corresponding algebraic version is also valid:

Theorem 2. — *The limit of a sequence of generic algebraic sets given by a finite number of formulas always exists and is an algebraic set that can be explicitly given as a Boolean expression involving the primitives of the additive forms of the formulas.*

In the complex case, the limit set of a family of algebraic sets given by a finite number of formulas has also an underlying semi-algebraic structure in the sense that it projects, by means of a rational map, onto a proper real semi-algebraic subset defined by expressions involving the absolute values of the primitives of the formulas. More precisely, we have the following result:

Theorem 3. — *The limit of a sequence of generic algebraic subsets given by a finite number of formulas with complex coefficients always exists; it is a subset with a complex structure obtained by means of a rational pull-back on semi-algebraic subsets defined explicitly in terms of Boolean expressions involving the absolute values of the primitives of the formulas.*

As an example of the situation in the complex case, we consider the following generalization of the $x^n + y^n = 1$ example given earlier. Let A_1, A_2, and P be complex polynomials. Then

$$\lim[A_1^n + A_2^n = P] = \partial([|A_1| < 1] \cap [|A_2| < 1] \cap [P \neq 0]) \cup [|A_1| = |A_2| > 1]).$$

This limit can be also understood as the pull-back by the polynomial map

$$(A_1, A_2)\colon \mathbb{C}^2 \to \mathbb{C}^2$$

of the Reinhardt preimage of the semi-algebraic subset of $\mathbb{R}^2$ given by the second member of the equation above, where the axes of $\mathbb{R}^2$ are taken as $|A_1|$ and $|A_2|$.

2. The real case

We start with the simplest cases and continue to more complicated cases until we reach general formulas in additive form. To simplify the exposition, we assume that *all semi-algebraic sets are generic* and we consider only formulas in which *all n-th powers are even.*

The simplest non-trivial formula of height 1 is $A^{2n} - P$, where A and P are real polynomials. We want to describe the limit of the algebraic subsets $[A^{2n} = P]$. As mentioned before, it is simpler to describe the limit of the semi-algebraic sets $\Omega_n = [A^{2n} \leq P]$. The strategy in the following lemma and in all subsequent lemmas in this section will be to give a candidate Ω for $\Omega_\infty = \lim \Omega_n$ and to show that $\Omega_\infty \subseteq \Omega$ and $\Omega \subseteq \Omega_\infty$, thus establishing that $\Omega_\infty = \Omega$.

All lemmas in this section say that the limit of a formula can be expressed as a Boolean combination of formulas of smaller height. Thus, they will provide a basis for proving Theorem 1 by induction on the height of the formula.

Lemma 2. — *Let A and P be polynomials. Then* $\lim[A^{2n} \leq P] = [A^2 \leq 1, P \geq 0]$.

Proof. — Let $\Omega_n = [A^{2n} \leq P]$, $\Omega_\infty = \lim \Omega_n$, and $\Omega = [A^2 \leq 1, P \geq 0]$. We shall show that $\Omega_\infty = \Omega$.

Take $z \in \Omega_\infty$. Then, by definition, there are sequences $z_n \to z$ and $k_n \to \infty$ with $z_n \in \Omega_{k_n}$, that is, $A(z_n)^{2k_n} \leq P(z_n)$. Since $A(z_n)^{2k_n} \geq 0$, we get $P(z_n) \geq 0$ and hence $P(z) = \lim P(z_n) \geq 0$. Moreover, the sequence $(P(z_n))$ is bounded and so $P(z_n) \leq L$ for some $L > 0$. This implies that $A(z_n)^2 \leq P(z_n)^{1/k_n} \leq L^{1/k_n}$. Therefore, $A(z)^2 = \lim A(z_n)^2 \leq \lim L^{1/k_n} = 1$. Hence, $z \in \Omega$.

Reciprocally, take $z \in \Omega$. Since Ω is generic, we have that $z = \lim z_n$, with $z_n \in [A^2 < 1, P > 0]$. From $A(z_n)^2 < 1$ we get that $A(z_n)^{2k} \to 0$ as $k \to \infty$. Since $P(z_n) > 0$, there is a k_n such that $A(z_n)^{2k_n} < P(z_n)$, that is, $z_n \in \Omega_{k_n}$. By increasing k_n beyond n if necessary to get $k_n \to \infty$, we conclude that $z \in \Omega_\infty$. □

The genericity hypothesis is essential to the lemma as stated. Although the proof shows that $\Omega_\infty \subseteq \Omega$ even when Ω is not generic, the reverse inclusion is not always true when Ω is not generic. The following example gives a taste of how things are more complicated in the general case. Let $A = y(y-1)^2 + 1$ and $P = x^2(x-1)$. Note that $[P \geq 0]$ is not the closure of $[P > 0]$ because $[P \geq 0]$ contains the line $[x = 0]$, which is not in the closure of $[P > 0]$ since P is negative around $x = 0$. Similarly, $[A^2 \leq 1]$ is not the closure of $[A^2 < 1]$ because of the line $[y = 1]$. As a consequence, $[A = 1, P \geq 0]$ is only partially contained in $\lim[A^n \leq P]$; only $[A = 1, P \geq 1]$ is part of the limit set. This example is typical of what happens in general: $\lim[A^{2n} \leq P]$ is equal to $[A^2 \leq 1, P \geq 0]$, except that $P \geq 1$ when $A = 1^+$, and $A = 1$ when $P = 0^-$.

The next lemma generalizes Lemma 2 and the $x^n + y^n = 1$ example given in §1:

Lemma 3. — *Let $A_1, \ldots, A_k$ and P be polynomials. Then*

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P] = \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P] = [A_1^2 \leq 1, \ldots, A_k^2 \leq 1, P \geq 0].$$

Proof. — Take $z \in \lim[A_1^{2n} + \cdots + A_k^{2n} \leq P]$. Then there are sequences $z_n \to z$ and $k_n \to \infty$ such that $A_i(z_n)^{2k_n} \leq A_1(z_n)^{2k_n} + \cdots + A_k(z_n)^{2k_n} \leq P(z_n)$. So

$$\lim[A_1^{2n} + \cdots + A_k^{2n} \leq P] \subseteq \bigcap_{i=1}^{k} \lim[A_i^{2n} \leq P] = \bigcap_{i=1}^{k} [A_i^2 \leq 1, P \geq 0],$$

by Lemma 2. Hence $z \in [A_1^2 \leq 1, \ldots, A_k^2 \leq 1, P \geq 0]$.