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SINGULARITIES OF LOGARITHMIC FOLIATIONS AND CONNECTEDNESS OF THE UNION OF LOGARITHMIC COMPONENTS

by

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To José Manuel Aroca on the occasion of his 60th birthday

Abstract. — We show that the singular locus of a logarithmic foliation may have components of any dimension between 0 and $n-2$. Moreover, it is shown that the union of the logarithmic components of the space of codimension one foliations of $\mathbb{P}^n_{\mathbb{C}}$, of fixed degree, is connected.

***Résumé* (Singularités des feuilletages logarithmiques et connectivité de l'union des composantes logarithmiques)**

On montre que le lieu singulier d'un feuilletage logarithmique peut avoir des composantes de dimension 1 à $n-2$. On montre aussi que, dans l'espace des feuilletages de codimension un de $\mathbb{P}^n_{\mathbb{C}}$, de degré fixé, la réunion des composantes logarithmiques est connexe.

1. Introduction

In this note we treat two simple questions.

The first is to explore some propoerties of the singular locus of a projective logarithmic foliation. Our motivation for this is the result given in [**6**] which states that, for a generic logarithmic foliation of $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, the singular scheme only exhibits components of dimensions 0 and $n-2$. We exploit this by considering non-generic logarithmic foliations, and by examining the possible dimensions of the components of their singular loci. It turns out that, for a non-generic such foliation, the singular locus may have components of any dimension between 0 and $n-2$, with the only constraint that the dimension $n-2$ is compulsory. On the other hand, we do not know if the isolated singularities, away from the polar divisors of the logarithmic form, which

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necessarily appear in case the foliation is generic (see Theorem 1.5 below), are persistent under a deformation into a foliation whose singular set admits components of dimensions ranging from 1 to $n-3$. In this direction however, see Example 2.6 at the end of this article.

The second consists in showing that the union of the logarithmic components of the space of codimension one foliations of $\mathbb{P}^n_{\mathbb{C}}$, of fixed degree, is connected. This is true also for the rational components of fixed degree. It is perhaps worth remarking that this result resembles a foundational contribution, by R. Hartshorne [**8**], on the connectedness of the Hilbert scheme parametrizing the family of closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ with a fixed Hilbert polynomial.

We start by recalling some definitions and (known) results.

A codimension one holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^n_{\mathbb{C}}$ is defined by an integrable polynomial 1-form $\omega = \sum_{i=0}^n A_i(z)\,dz_i$, where each A_i is a homogeneous polynomial, say of degree $k-1$, and such that ω contracts to zero by the radial vector field $R = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$, that is, $\iota_R\,\omega = \sum_{i=0}^n z_i A_i(z) \equiv 0$.

Such 1-forms define global sections of $\Omega^1_{\mathbb{P}^n_{\mathbb{C}}}(k)$ and, since $\omega \wedge d\omega \equiv 0$, they make up a Zariski closed subset of $\mathbb{P}(H^0(\Omega^1_{\mathbb{P}^n_{\mathbb{C}}}(k)))$.

The degree of a foliation $\mathcal{F}$, $\deg \mathcal{F}$, is the number of tangencies of the leaves of $\mathcal{F}$ with a generic one-dimensional linear subspace of $\mathbb{P}^n_{\mathbb{C}}$. A simple calculation shows that $\deg \mathcal{F} = k-2$ if the 1-form defining $\mathcal{F}$ has components of degree $k-1$. We denote by $Fol(\mathbb{P}^n_{\mathbb{C}}; k)$ the space of codimension one holomorphic foliations of degree $k-2$ of $\mathbb{P}^n_{\mathbb{C}}$.

A very interesting question is that of recognizing the irreducible components of $Fol(\mathbb{P}^n_{\mathbb{C}}; k)$ and to determine their dimensions and degrees. Results in this direction can be found in [**4**], [**5**] and [**3**].

Logarithmic foliations are defined as follows: given positive integers $d_0, \dots, d_m$, set $\underline{\mathrm{d}} = d_0, \dots, d_m$ and $\mathrm{d} = \sum_{i=0}^m d_i$. Consider the hyperplane

$$\mathbb{P}(m-1, \mathrm{d}) = \{(\lambda_0, \dots, \lambda_m) \in \mathbb{P}^m_{\mathbb{C}} \;:\; \sum_{i=0}^m d_i\,\lambda_i = 0\}. \tag{1}$$

Define a rational map Ψ by

$$\begin{aligned} \mathbb{P}(m-1, \mathrm{d}) \times \prod_{i=0}^m \mathbb{P}\left(H^0(\mathbb{P}^n_{\mathbb{C}}; \mathcal{O}(d_i))\right) &\xrightarrow{\ \Psi\ } Fol(\mathbb{P}^n_{\mathbb{C}}; \mathrm{d}) \\ ((\lambda_0, \dots, \lambda_m), (F_0, \dots, F_m)) &\longmapsto \left(\prod_{j=0}^m F_j\right) \sum_{i=0}^m \lambda_i \frac{dF_i}{F_i}. \end{aligned} \tag{2}$$

Note that $\iota_R \left(\prod_{j=0}^m F_j\right) \sum_{i=0}^m \lambda_i \frac{dF_i}{F_i} = \left(\prod_{j=0}^m F_j\right) \sum_{i=0}^m d_i\,\lambda_i \equiv 0$. Also, it's immediate that this form is closed (hence integrable).

The closure of the image of Ψ is the set $\mathrm{Log}_n(d_0, \dots, d_m)$ of logarithmic foliations of type $\underline{\mathrm{d}}$ and degree $\mathrm{d}-2$ of $\mathbb{P}^n_{\mathbb{C}}$.

Let us recall the following result, due to Calvo-Andrade [**2**]:

Theorem 1.1. — *For fixed* $\underline{\mathrm{d}}$ *and* $n \geq 3$, $\mathrm{Log}_n(d_0, \ldots, d_m)$ *is an irreducible component of* $Fol(\mathbb{P}^n_{\mathbb{C}}; \mathrm{d})$.

Remark 1.2. — In case $m = 1$, $\mathrm{Log}_n(d_0, d_1)$ is a component of $Fol(\mathbb{P}^n_{\mathbb{C}}; d_0 + d_1)$, called a *rational* component. This is because such $\mathcal{F}$'s are necessarily given by $\varpi = d_1 F_1\, dF_0 - d_0 F_0\, dF_1$, and they all admit a rational first integral, namely, $\dfrac{F_0^{d_1}}{F_1^{d_0}}$.

Let $\mathcal{F}$ be a logarithmic foliation of $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, given, in $\mathbb{C}^{n+1}$, by the 1-form

$$\omega = \left(\prod_{j=0}^{m} F_j\right) \sum_{i=0}^{m} \lambda_i \frac{dF_i}{F_i} = \lambda_0 \widehat{F_0}\, dF_0 + \cdots + \lambda_m \widehat{F_m}\, dF_m. \tag{3}$$

where $F_i \in \mathbb{C}[z_0, \ldots, z_n]$ is homogeneous of degree $d_i \geq 1$, $\sum\limits_{i=0}^{m} d_i = \mathrm{d}$, $\sum\limits_{i=0}^{m} \lambda_i\, d_i = 0$ and $\widehat{F_j} = F_1 \cdots F_{j-1} F_{j+1} \cdots F_m$.

The singular set of $\mathcal{F}$ is $S(\mathcal{F}) = \{z \in \mathbb{P}^n_{\mathbb{C}} : \omega(z) = 0\}$ and the fact that ω is integrable imposes the existence of codimension 2 components in $S(\mathcal{F})$ (see [**9**]).

It is immediate that $S(\mathcal{F})$ contains the union of all codimension two subsets $F_i = F_j = 0$.

Let $D_i = \{p \in \mathbb{P}^n_{\mathbb{C}} : F_i(p) = 0\}$ be the divisor associated to F_i.

Definition 1.3. — *A logarithmic foliation given by* $\omega = \lambda_0 \widehat{F_0}\, dF_0 + \cdots + \lambda_m \widehat{F_m}\, dF_m$ *is said to be generic if*

$$\begin{cases} F_i,\ i = 0, \ldots, m, & \textit{is irreducible} \\ \textit{the } D_i\text{'s},\ i = 0, \ldots, m, & \textit{are smooth and in general position.} \\ \lambda_i \neq 0,\ i = 0, \ldots, m. & \end{cases} \tag{4}$$

Remark that (4) defines a Zariski open subset of

$$\mathbb{P}(m-1, \mathrm{d}) \times \prod_{i=0}^{m} \mathbb{P}\left(H^0(\mathbb{P}^n_{\mathbb{C}}; \mathcal{O}(d_i))\right).$$

The following result was proven in [**6**]:

Proposition 1.4. — *Let* $\mathcal{F} \in \mathrm{Log}_n(d_0, \ldots, d_m)$ *be given by* $\omega = \left(\prod\limits_{j=0}^{m} F_j\right) \sum\limits_{i=0}^{m} \lambda_i \dfrac{dF_i}{F_i}$ *and assume* $\mathcal{F}$ *is generic. Then* $S(\mathcal{F})$ *has only codimension 2 and dimension 0 components.*

Proof. — The proof of this is a simple argument, which we present here for the sake of completness:

We will show that, if a point is non isolated in $S(\mathcal{F})$, then it lies in $D_i \cap D_j$ for some $i < j$. Indeed, let C be an irreducible component of $S(\mathcal{F})$ of dimension $1 \leq \dim C \leq n-2$. By ampleness and general position, we may pick a point $p \in C$

lying in the intersection of precisely k of the divisors D_i, $1 \leq k \leq \min\{n, m+1\}$. Let f_i be a local equation for D_i at p. Near p, the foliation $\mathcal{F}$ is given by the 1–form

$$\varpi = f_0 \dots f_m \sum_{i=0}^{m} \lambda_i \frac{df_i}{f_i}.$$

Renumbering the indices we may assume $p \in D_0 \cap \cdots \cap D_{k-1}$. The local defining equations $f_i = 0$ of the D_i's, for $i = 0, \dots, k-1$, are part of a regular system of parameters, i.e., $df_0, \dots, df_{k-1}$ are linearly independent at p. Write $\tilde{g} = f_k \cdots f_m$. Since $p \notin D_j$, $k \leq j \leq m$, we may assume $\tilde{g}$ vanishes nowhere around p and write ϖ as

$$\varpi = f_0 \cdots f_{k-1}\, \tilde{g} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \sum_{i=k}^{m} \lambda_i \frac{df_i}{f_i}\right] = f_0 \cdots f_{k-1}\, \tilde{g} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \eta\right],$$

where $\eta = \sum_{i=k}^{m} \lambda_i \frac{df_i}{f_i}$ is a holomorphic closed form near p. Since η is closed, it is exact near p, say $\eta = d\xi$. Set $\vartheta = \varpi/\tilde{g}$. Then $\mathcal{F}$ is defined around p by

$$\vartheta = f_0 \cdots f_{k-1} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + d\xi\right] = f_0 \cdots f_{k-1} \left[\lambda_0 \frac{d(\exp[\xi/\lambda_0] f_0)}{\exp[\xi/\lambda_0] f_0} + \sum_{j=1}^{k-1} \lambda_j \frac{df_j}{f_j}\right].$$

Set $z_0 = \exp[\xi/\lambda_0] f_0$ and $z_1 = f_1, \dots, z_{k-1} = f_{k-1}$. Since $u = \exp[\xi/\lambda_0]$ is a unit, we have that also $z_0, \dots, z_{k-1}$ are part of a regular system of parameters at p. Now ϑ can be written as

$$\vartheta = \frac{z_0}{u}\, z_1 \cdots z_{k-1} \left[\lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1} \lambda_j \frac{dz_j}{z_j}\right].$$

Thus $\mathcal{F}$ is defined around p by the 1-form

$$\widetilde{\vartheta} = z_0\, z_1 \cdots z_{k-1} \left[\lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1} \lambda_j \frac{dz_j}{z_j}\right] = \sum_{j=0}^{k-1} \lambda_j\, z_0 \cdots \widehat{z_j} \cdots z_{k-1}\, dz_j.$$

If $k = 1$, the above expression shows that the foliation is defined near p by dz_0 and then is non-singular at p. Hence we necessarily have $k \geq 2$. Note that the ideal of the scheme of zeros of $\widetilde{\vartheta}$ (as well as of ω) near p is generated by the k monomials $z_0 \cdots \widehat{z_j} \cdots z_{k-1}$ with $0 \leq j \leq k-1$. That is just the scheme union $\cup_{i,j} D_i \cap D_j$, for $0 \leq i < j \leq k-1$. Thus C must be contained in $D_i \cap D_j$, for some $i < j$, and therefore C is an irreducible component of $D_i \cap D_j$ and $\dim C = n-2$. □

The main result of [**6**] is:

Theorem 1.5. — *Let $\mathcal{F}$ be a generic logarithmic foliation of $\mathbb{P}^n_{\mathbb{C}}$ of type $\underline{\mathrm{d}} = d_0, \dots, d_m$, given by $\lambda_0 \widehat{F_0}\, dF_0 + \cdots + \lambda_m \widehat{F_m}\, dF_m$. Then the singular scheme $S(\mathcal{F})$ of $\mathcal{F}$ can be written as a disjoint union*

$$S(\mathcal{F}) = Z \bigcup R \tag{5}$$

where $Z = \bigcup_{i<j} D_{ij}$ *and* R *is finite, consisting of*

$$N(n, \underline{\mathrm{d}}) = \textit{the coefficient of } h^n \textit{ in } \frac{(1-h)^{n+1}}{\prod_{i=0}^{m}(1-d_i h)} \tag{6}$$

points counted with natural multiplicities.

It turns out that

$$N(n, \underline{\mathrm{d}}) = \sum_{j=0}^{n} \binom{m-1}{j} \mathscr{W}_{n-j}(d_0 - 1, \ldots, d_m - 1), \tag{7}$$

where $\mathscr{W}_k(X_0, \ldots, X_m)$ is the complete symmetric function of degree k in $m+1$ variables, that is, $\mathscr{W}_0 = 1$ and $\mathscr{W}_k(X_0, \ldots, X_m) = \sum_{i_0+\cdots+i_m=k} X_0^{i_0} \cdots X_m^{i_m}$.

It follows that, whenever at least one $d_i \geq 2$, we have $N(n, \underline{\mathrm{d}}) > 0$. Remark also that by (6) or (7), in case all $d_i = 1$, only for $n \geq m$ we have $N(n, \underline{\mathrm{d}}) = 0$.

Proof. — The proof of this theorem is based on the fact that, if $\mathscr{F}$ is a generic logarithmic foliation, then the codimension two part of its singular scheme is equal to the singular scheme of the normal crossings divisor $\bigcup_i D_i$. We can then use Aluffi's formula ([**1**]) for the Segre class of $\bigcup_i D_i$. □

2. Deformations of generic logarithmic foliations

We start by stating some simple remarks, in the form we need. The first one is given in Fulton ([**7**], 8.4.13):

Remark 2.1. — *The hypersurfaces of degree* d_j *in* $\mathbb{P}^n_{\mathbb{C}}$ *are parametrized by* $\mathbb{P}^{M_j}_{\mathbb{C}}$, *where* $M_j = \binom{d_j+n}{n} - 1$. *Let* $X = (X_0 : \ldots : X_n)$ *be homogeneous coordinates in* $\mathbb{P}^n_{\mathbb{C}}$, $\mu^j_{(\ell)}$ *homogeneous coordinates in* $\mathbb{P}^{M_j}_{\mathbb{C}}$ *and* $\Phi_{d_j} = \sum \mu^j_{(\ell)} X^{(\ell)}$ *the expression defining the universal hypersurface of degree* d_j *in* $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}}$. *The incidence variety* $\mathscr{V} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}}$ *is defined by*

$$\mathscr{V} = \{(X, \mu^j) \, : \, \Phi_{d_j}(\mu^j, X) = 0\}.$$

$\mathscr{V}$ *is a smooth, irreducible (hence connected) subvariety of* $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}}$ *of codimension* 1. *Let* $\pi : \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^{M_j}_{\mathbb{C}} \longrightarrow \mathbb{P}^{M_j}_{\mathbb{C}}$ *be the projection and consider its restriction to* $\mathscr{V}$, $\pi_{|\mathscr{V}} : \mathscr{V} \longrightarrow \mathbb{P}^{M_j}_{\mathbb{C}}$. *The fiber* D_{μ^j} *of* $\pi_{|\mathscr{V}}$ *over* μ^j *is the corresponding hypersurface. Notice that the generic fiber is an irreducible, non-singular hypersurface of degree* d_j.

Next we have

Remark 2.2. — *An irreducible hypersurface of degree* $d \geq 2$ *in* $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$, *may have a singular set of dimension* $0, 1, \ldots, n-2$. *Examples of such are given by:*

$$P_k(z_0, \ldots, z_n) = z_0^d + z_0^{d-1}(z_1 + \cdots + z_k) + z_{k+1}^d + \cdots + z_n^d$$