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ON THE EFFACEABILITY OF CERTAIN δ-FUNCTORS

by

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Abstract. — We prove a conjecture of the first author for $\mathrm{GL}_2(F)$, where F is a finite extension of $\mathbb{Q}_p$.

***Résumé* (Sur l'effaçabilité de certains δ-foncteurs).** — On démontre une conjecture du premier auteur pour $\mathrm{GL}_2(F)$, où F est une extension finie de $\mathbb{Q}_p$.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ and let $\mathfrak{o}$ be its ring of integers. Let $G := \mathrm{GL}_2(F)$, let $K := \mathrm{GL}_2(\mathfrak{o})$, and let Z be the centre of G. Let A be a finite local Artinian $\mathbb{Z}_p$-algebra with residue field k (necessarily finite, of characteristic p). Recall that a representation V of G on an A-module is said to be *smooth* if for all $v \in V$ the stabilizer of v is an open subgroup of G. Let $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ denote the category of smooth G-representations. Further recall that a smooth G-representation V is *admissible* if for every open subgroup J of G the space V^J of J-invariants is a finite A-module. Let $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of admissible representations. The categories $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ and $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ are abelian. In practice, one is interested in admissible representations, but $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ does not have enough injectives. The category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ has enough injectives, but it is too big. To remedy this the first author, in **[2]**, **[3]**, has introduced an intermediate category of locally admissible representations $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$. We recall the definition: If V is a smooth A-representation of G, a vector $v \in V$ is called *locally admissible* if the $A[G]$-submodule of V generated by v is admissible; a smooth representation V of G over A is then called *locally admissible* if every $v \in V$ is locally admissible. We let $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ denote the full subcategory of

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$\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of locally admissible representations. The category $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ is abelian and has enough injectives [**2**, Prop. 2.2.18], [**3**, Prop. 2.1.1].

We introduce some variants of the preceding categories:

If $\zeta : Z \to A^\times$ is a smooth character, then we denote by $\mathrm{Mod}_{G,\zeta}^{\mathrm{adm}}(A)$, $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$, and $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(A)$ the full subcategories of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$, and $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ respectively, consisting of representations admitting ζ as a central character. We also let $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$ denote the full subcategory of $\mathrm{Mod}_K^{\mathrm{sm}}(A)$ consisting of K-representations admitting $\zeta_{|Z\cap K}$ as a central character. The categories $\mathrm{Mod}_{G,\zeta}^{\mathrm{adm}}(A)$, $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$, $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(A)$, and $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$ are abelian, and the last three have enough injectives. (See Lemma 2.4 below.)

In this note we show that the restriction to K of an injective object in $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$) is an injective object in $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(A)$ (resp. $\mathrm{Mod}_K^{\mathrm{sm}}(A)$). This implies that certain δ-functors defined in [**3**] are effaceable, and remain effaceable when restricted to $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$. In particular, it proves Conjecture 3.7.2 of [**3**] for $\mathrm{GL}_2(F)$.

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2. Injectives

We establish some simple results about injective objects in various contexts. In this section we change our notational conventions from those of the introduction, and let G denote an arbitrary p-adic analytic group. We let $\mathfrak{m}$ denote the maximal ideal of the ring of coefficients A.

2.1. Lemma. — *If G is compact, if V is an injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(k)$, and if W is an injective envelope of V in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, then the inclusion $V \hookrightarrow W$ induces an isomorphism $V \xrightarrow{\sim} W[\mathfrak{m}]$.*

Proof. — Certainly the inclusion $V \hookrightarrow W$ factors through an inclusion $V \hookrightarrow W[\mathfrak{m}]$. Since the source is injective, this inclusion splits. If C denotes a complement to the inclusion, then $V \cap C = 0$, and thus $C = 0$ (as W is an essential extension of V). This proves the lemma. □

2.2. Lemma. — *Let H be a finite index open subgroup of G.*

(i) *An object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ is admissible (resp. locally admissible) as a G-representation if and only if it is so as an H-representation.*

(ii) *If V is an object of* $\mathrm{Mod}_H^{\mathrm{sm}}(A)$*, so that* $\mathrm{Ind}_H^G V$ ($\xrightarrow{\sim} A[G]\otimes_{A[H]} V$) *is an object of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$*, then* $\mathrm{Ind}_H^G V$ *is admissible (resp. locally admissible) as a G-representation if and only if V is admissible (resp. locally admissible) as an H-representation.*

Proof. — The admissibility claim of part (i) is clear, since H contains a cofinal collection of open subgroups of G. Since H has finite index in G, the group ring $A[G]$ is finitely generated as an $A[H]$-module, and thus an $A[G]$-module is finitely generated if and only if it is finitely generated as an $A[H]$-module. The local admissibility claim of part (i) follows from this, together with the admissibility claim, since an $A[G]$-module (resp. $A[H]$-module) is locally admissible if and only if every finitely generated submodule is admissible.

To prove the if direction of claim (ii), suppose first that V is an admissible H-representation. If we write G as a union of finitely many left H-cosets, say $G=\bigcup_{i=1}^n g_iH$, if H' is an open subgroup of H, and if we write $H'':=H'\cap\bigcap_{i=1}^n g_iHg_i^{-1}$, then

$$(\mathrm{Ind}_H^G V)^{H'}\subset(\mathrm{Ind}_H^G V)^{H''}\xrightarrow{\sim}(A[G]\otimes_{A[H]} V)^{H''}\xrightarrow{\sim}\oplus_{i=1}^n(g_iV)^{H''}=\oplus_{i=1}^n g_iV^{g_i^{-1}H''g_i}.$$

Since $g_i^{-1}H''g_i$ is an open subgroup of H, each of the summands appearing on the right-hand side is a finite A-module, and thus so is their direct sum. Thus $\mathrm{Ind}_H^G V$ is admissible as claimed. If we suppose that V instead is locally admissible, or equivalently, is the inductive limit of its admissible subrepresentations, we see that the same is true of $\mathrm{Ind}_H^G V$, since Ind_H^G commutes with the formation of induction limits (being naturally isomorphic to $A[G]\otimes_{A[H]}-$).

To prove the other direction of (ii), note first that the inclusion $A[H]\subset A[G]$ gives rise to an H-equivariant embedding $V\hookrightarrow A[G]\otimes_{A[H]}V\xrightarrow{\sim}\mathrm{Ind}_H^G V$. Thus if $\mathrm{Ind}_H^G V$ is (locally) admissible as a G-representation, and hence also (locally) admissible as an H-representation, by part (i), the same is true of its H-subrepresentation V. □

2.3. Definition. — If Z denotes the centre of G, if $\zeta: Z\to A^\times$ is a smooth character and V is a representation of G over A, then we let

$$V^{Z=\zeta}:=\{v\in V\mid z\cdot v=\zeta(z)v\text{ for all }z\in Z\}.$$

Since the subrepresentation of a smooth admissible (resp. smooth locally admissible, resp. smooth) representation is again smooth admissible (resp. smooth locally admissible, resp. smooth), we see, in the context of the preceding definition, that the construction $V\mapsto V^{Z=\zeta}$ induces a functor $\mathrm{Mod}_G^{\mathrm{adm}}(A)\to\mathrm{Mod}_{G,\zeta}^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)\to\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(A)$, resp. $\mathrm{Mod}_G^{\mathrm{sm}}(A)\to\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(A)$) that is right adjoint to the forgetful functor. In particular, the functor $V\mapsto V^{Z=\zeta}$ preserves injectives.

2.4. Lemma. — *If $\zeta : Z \to A^\times$ is a smooth character, then each of the categories* $\mathrm{Mod}^{\mathrm{adm}}_{G,\zeta}(A)$, $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$, *and* $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$ *are abelian, and the last two have enough injectives.*

Proof. — The abelianess claims are evident. To establish the claim regarding injectives, let V be an object of $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$ (resp. $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$) and let $V \hookrightarrow W$ be an $A[G]$-linear embedding of V into an injective object in $\mathrm{Mod}^{\mathrm{l.adm}}_{G}(A)$ (resp. $\mathrm{Mod}^{\mathrm{sm}}_{G}(A)$). This embedding then factors through an embedding $V \hookrightarrow W^{Z=\zeta}$, and the latter object is injective in $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(A)$ (resp. $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$), as was noted above. □

2.5. Lemma. — *Let G be a compact p-adic analytic group, let H be a closed subgroup containing the centre of G and let $\zeta : Z \to A^\times$ be a smooth character. If V is injective in* $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$, *then it is also injective in* $\mathrm{Mod}^{\mathrm{sm}}_{H,\zeta}(A)$.

Proof. — Let $\iota : V \hookrightarrow J$ be an injective envelope of V in $\mathrm{Mod}^{\mathrm{sm}}_{G}(A)$. Since V is injective in $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(A)$ and ι is essential we deduce that $\iota(V) = J^{Z=\zeta}$. Proposition 2.1.11 in **[3]** implies that J is injective in $\mathrm{Mod}^{\mathrm{sm}}_{H}(A)$ and thus $J^{Z=\zeta}$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_{H,\zeta}(A)$. □

3. Main result

We introduce notation for some subgroups of $G := \mathrm{GL}_2(F)$ that we will need to consider, namely: we write $G^+ := \{g \in G : \mathrm{val}_F(\det g) \equiv 0 \pmod 2\}$ and $G^0 := \{g \in G : \mathrm{val}_F(\det g) = 0\}$, write $I := \left(\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \varpi\mathfrak{o} & \mathfrak{o}^\times \end{smallmatrix}\right)$ (an Iwahori subgroup of K) and let I_1 denote the maximal pro-p subgroup of I, let $N_G(I)$ denote the normalizer in G of I, set $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right) \in N_G(I)$, and write $N_0 := \left(\begin{smallmatrix} 1 & \mathfrak{o} \\ 0 & 1 \end{smallmatrix}\right)$.

3.1. Lemma. — *If $\iota : V \hookrightarrow J$ is an injective envelope of V in* $\mathrm{Mod}^{\mathrm{sm}}_{I}(A)$, *then any isomorphism $\psi : V \xrightarrow{\cong} V^\Pi$ extends to an isomorphism $J \cong J^\Pi$.*

Proof. — Since $\iota^\Pi : V^\Pi \hookrightarrow J^\Pi$ is an injective envelope of V^Π in $\mathrm{Mod}^{\mathrm{sm}}_{I}(A)$, the assertion follows from the fact that injective envelopes are unique up to isomorphism. □

3.2. Lemma. — *For an injective admissible object J in* $\mathrm{Mod}^{\mathrm{sm}}_{I}(A)$ *the following are equivalent:*

(i) $J \cong J^\Pi$;
(ii) $J[\mathfrak{m}]^{I_1} \cong (J[\mathfrak{m}]^{I_1})^\Pi$;
(iii) $\dim_k \mathrm{Hom}_I(\chi, J[\mathfrak{m}]^{I_1}) = \dim_k \mathrm{Hom}_I(\chi^\Pi, J[\mathfrak{m}]^{I_1})$, $\forall \chi \in \mathrm{Irr}_I(k)$.

Proof. — Since $J[\mathfrak{m}]^{I_1} \hookrightarrow J$ is essential the equivalence of (i) and (ii) follows from Lemma 3.1. Since J is admissible $J[\mathfrak{m}]^{I_1}$ is a finite dimensional k-vector space. Since the order of I/I_1 is prime to p we may write $J[\mathfrak{m}]^{I_1} \cong \oplus_{\chi\in \mathrm{Irr}_I(k)}\chi^{\oplus m_\chi}$ and thus $J[\mathfrak{m}]^{I_1} \cong (J[\mathfrak{m}]^{I_1})^{\Pi}$ if and only if $m_\chi = m_{\chi^\Pi}$. Hence, (ii) is equivalent to (iii). □

3.3. Lemma. — *If J is an admissible injective object in $\mathrm{Mod}^{\mathrm{sm}}_K(A)$, then*

$$\dim_k \mathrm{Hom}_I(\chi, J[\mathfrak{m}]^{I_1}) = \dim_k \mathrm{Hom}_I(\chi^{\Pi}, J[\mathfrak{m}]^{I_1}), \quad \forall \chi \in \mathrm{Irr}_I(k).$$

Proof. — Since $J[\mathfrak{m}]$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_K(k)$ we may assume that $A = k$ so that $J[\mathfrak{m}] = J$. Further, it is enough to prove the statement for $J = \mathrm{Inj}\,\sigma$ an injective envelope of an irreducible K-representation σ, since any admissible injective object of $\mathrm{Mod}^{\mathrm{sm}}_K(A)$ is isomorphic to a finite direct sum of such representations. If $k = \overline{\mathbb{F}}_p$ then the assertion for $J = \mathrm{Inj}\,\sigma$ follows from [**4**, Lem. 6.4.1, 4.2.19, 4.2.20], see also the proof of [**1**, Lem. 9.6]. (It is enough to assume that k contains the residue field of F, in which case every irreducible k-representation of K or I is absolutely irreducible.) The result for general k follows by Galois descent. □

3.4. Theorem. — *If V is an object in $\mathrm{Mod}^{\mathrm{adm}}_{G^0}(A)$ such that $V \cong V^{\Pi}$, then there exists a G^0-equivariant injection $V \hookrightarrow \Omega$ in $\mathrm{Mod}^{\mathrm{adm}}_{G^0}(A)$ such that $V|_K \hookrightarrow \Omega|_K$ is an injective envelope of $V|_K$ in $\mathrm{Mod}^{\mathrm{sm}}_K(A)$.*

Proof. — The proof is a variation on constructions of [**1**] and [**5**]. It relies on the fact that G^0 is an amalgam of K and K^{Π} along $I = K \cap K^{\Pi}$. Let $\iota_0 : V|_K \hookrightarrow J_0$ be an injective envelope of V in $\mathrm{Mod}^{\mathrm{sm}}_K(A)$ and let $\iota_1 : V|_{K^{\Pi}} \hookrightarrow J_1$ be an injective envelope of V in $\mathrm{Mod}^{\mathrm{sm}}_{K^{\Pi}}(A)$. We claim that there exists an I-equivariant isomorphism $\varphi : J_0 \xrightarrow{\cong} J_1$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\iota_0} & J_0 \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong\,\varphi} \\ V & \xrightarrow{\iota_1} & J_1. \end{array}$$

commutes. Granting the claim we may using φ transport the action of K^{Π} on J_0 such that the two actions of I on J_0 via embeddings $I \hookrightarrow K$, $I \hookrightarrow K^{\Pi}$ coincide. Since G^0 is an amalgam of K and K^{Π} along $I = K \cap K^{\Pi}$ we obtain an action of G^0 on J_0 and since the diagram is commutative $\iota_0 : V \hookrightarrow J_0$ is G^0-equivariant.

To prove the claim we closely follow the proof of Theorem 9.8 [**1**]. Since I is an open subgroup of K, $J_0|_I$ is an injective object in $\mathrm{Mod}^{\mathrm{sm}}_I(A)$ and thus there exists an idempotent $e \in \mathrm{End}_{A[I]}(J_0)$ such that $e \circ \iota_0 = \iota_0$ and $\iota_0 : V \hookrightarrow eJ_0$ is an injective envelope of V in $\mathrm{Mod}^{\mathrm{sm}}_I(A)$. By Lemma 3.1 there exists an isomorphism $\beta : eJ_0 \xrightarrow{\cong} (eJ_0)^{\Pi}$ extending the given isomorphism $\alpha : V \xrightarrow{\cong} V^{\Pi}$. Lemma 3.2 implies that

$$\dim_k \mathrm{Hom}_I(\chi, eJ_0[\mathfrak{m}]^{I_1}) = \dim_k \mathrm{Hom}_I(\chi^{\Pi}, eJ_0[\mathfrak{m}]^{I_1}), \quad \forall \chi \in \mathrm{Irr}_I(k). \tag{3.5}$$