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DEFORMATIONS OF $G_{\mathbb{Q}_p}$ AND $\mathrm{GL}_2(\mathbb{Q}_p)$ REPRESENTATIONS

by

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Abstract. — We show that Colmez's functor from $\mathrm{GL}_2(\mathbb{Q}_p)$ representations to $G_{\mathbb{Q}_p}$ representation produces essentially all two dimensional representations of $G_{\mathbb{Q}_p}$. The method compares the deformation theory for the two kinds of representations: An Ext group calculation of Colmez implies that the deformation space for $\mathrm{GL}_2(\mathbb{Q}_p)$ representations is closed in that for $G_{\mathbb{Q}_p}$-representations. A local version of the Gouvêa-Mazur "infinite fern" argument shows that this closed subspace is also dense.

***Résumé* (Déformations de $G_{\mathbb{Q}_p}$ et représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$).** — On montre que le foncteur de Colmez, entre les représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et celles de $G_{\mathbb{Q}_p}$, produit essentiellement toutes les représentations bidimensionnelles de $G_{\mathbb{Q}_p}$. Notre méthode compare les théories de déformation des deux types de représentations : un calcul de groupe Ext effectué par Colmez implique que l'espace de déformation pour les représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ est fermé dans celui des $G_{\mathbb{Q}_p}$-représentations. Une version locale de l'argument «infinite fern» de Gouvêa-Mazur montre que ce sous-espace fermé est également dense.

Introduction

The purpose of this appendix is to prove that Colmez's functor $\mathbf{V}$ from $\mathrm{GL}_2(\mathbb{Q}_p)$-representations to $G_{\mathbb{Q}_p}$-representations produces essentially all two dimensional representations of $G_{\mathbb{Q}_p}$. Here $G_{\mathbb{Q}_p}$ denotes the absolute Galois group of $\mathbb{Q}_p$. More precisely, let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathscr{O}$ and uniformiser $\pi_{\mathscr{O}}$. For a continuous representation of $G_{\mathbb{Q}_p}$ on a 2-dimensional E-vector space V, and $L \subset V$ a $G_{\mathbb{Q}_p}$-stable $\mathscr{O}$-lattice, we denote by $\bar{V}$ the semi-simplification of $L/\pi_{\mathscr{O}}L$. This does not depend on L.

We denote by $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ the cyclotomic character and by $\omega : G_{\mathbb{Q}_p} \to \mathbb{F}_p^\times$ its mod p reduction. Finally we denote by ω_2 a fundamental character of level 2 of $I_{\mathbb{Q}_p}$.

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Then our main result is the following

Theorem 0.1. — *Suppose that $p > 2$ and if $p = 3$ assume that $\bar{V}$ is not of the form $\left(\begin{smallmatrix}\omega & 0\\ 0 & 1\end{smallmatrix}\right) \otimes \chi$ and $\bar{V}|_{I_{\mathbb{Q}_p}}$ is not of the form $\begin{pmatrix}\omega_2^2 & 0\\ 0 & \omega_2^6\end{pmatrix} \otimes \chi$.*

1. *If V is irreducible, then there exists an admissible $\mathscr{O}$-lattice Π with central character $\det V \cdot \chi_{\mathrm{cyc}}^{-1}$ such that*
$$\mathbf{V}(\Pi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V.$$
2. *If $\bar{V} \not\sim \left(\begin{smallmatrix}1 & 0\\ 0 & \omega\end{smallmatrix}\right) \otimes \chi$ then for any $G_{\mathbb{Q}_p}$-stable $\mathscr{O}$-lattice $L \subset V$, then there exists an admissible $\mathscr{O}$-lattice Π with central character $\det V \cdot \chi_{\mathrm{cyc}}^{-1}$ such that $\mathbf{V}(\Pi) \xrightarrow{\sim} L$.*

Here by an admissible $\mathscr{O}$-lattice we mean a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a p-torsion free, p-adically complete and separated $\mathscr{O}$-module Π such that for $n \geq 1$ the quotient $\Pi/p^n\Pi$ is a smooth, finite length representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. These are exactly the representations to which Colmez's functor applies, and we can then extend it to admissible $\mathscr{O}$-lattices, so that $\mathbf{V}(\Pi)$ is a $G_{\mathbb{Q}_p}$-stable $\mathscr{O}$-lattice in V.

To explain the idea of the argument, let $\mathbb{F}$ be the residue field of $\mathscr{O}$ and $V_{\mathbb{F}}$ a two dimensional representation of $G_{\mathbb{Q}_p}$. Suppose that $V_{\mathbb{F}} \not\sim \left(\begin{smallmatrix}1 & *\\ 0 & \omega\end{smallmatrix}\right) \otimes \chi$. Then Colmez shows that there is a smooth, finite length representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an $\mathbb{F}$-vector space $\bar{\pi}$, having central character $\bar{\psi} = \det V_{\mathbb{F}} \chi_{\mathrm{cyc}}^{-1}$ and such that $\mathbf{V}(\bar{\pi}) \xrightarrow{\sim} V_{\mathbb{F}}$.

Now fix a continuous character $\psi : G_{\mathbb{Q}_p} \to \mathscr{O}^{\times}$ lifting $\bar{\psi}$. For simplicity of notation we will assume that $V_{\mathbb{F}}$, and hence $\bar{\pi}$ has only scalar endomorphisms.(1) Then one can define three deformation problems over the category of finite, local, Artinian $\mathscr{O}$-algebras: The first one, $D_{\bar{\pi},\psi}$, parameterizes deformations of $\bar{\pi}$ with central character ψ. The other two $D_{V_{\mathbb{F}}}$, (resp. $D_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$) parameterize deformations of $V_{\mathbb{F}}$ (resp. deformations of $V_{\mathbb{F}}$ with determinant $\psi\chi_{\mathrm{cyc}}$). Each of these deformation problems is pro-representable by a complete local $\mathscr{O}$-algebra which we denote by $R_{\bar{\pi},\psi}$, $R_{V_{\mathbb{F}}}$ and $R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ respectively. Colmez's functor produces a map

$$\operatorname{Spec} R_{\bar{\pi},\psi} \to \operatorname{Spec} R_{V_{\mathbb{F}}} \tag{1}$$

and one of the main results of [**9**, § VII] is that (1) induces an injection on tangent spaces. Hence it is a closed embedding.

One can sometimes show that this embedding factors through $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$, but this does not always hold.(2) On the other hand, results of Colmez and Berger-Breuil allow one to show that any crystalline point with distinct Hodge-Tate weights and determinant $\psi\chi_{\mathrm{cyc}}$ is in the image of (1). By imitating the "infinite fern" argument of Gouvêa-Mazur [**10**], we are able to show that the set of crystalline points is dense in $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}[1/p]$:

(1) Below this condition will be avoided by using framings.
(2) However see the remark at the end of this introduction.

Theorem 0.2. — *Suppose that $p > 2$, and that $V_{\mathbb{F}} \not\sim \left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right)$ and $V_{\mathbb{F}}|_{I_{\mathbb{Q}_p}} \not\sim \left(\begin{smallmatrix} \omega^2 & 0 \\ 0 & \omega_2^6 \end{smallmatrix}\right)$. Then the set of closed points $x \in \operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}[1/p]$ such that the corresponding $G_{\mathbb{Q}_p}$-representation is crystalline is Zariski dense.*

In fact it is technically simpler to work with crystalline points satisfying some mild non-degeneracy conditions, so the results in the text refer to "benign" or "twisted benign" points. Such points are, in particular, crystalline.

As an immediate consequence of 0.2, one sees that the image of (1) contains $\operatorname{Spec} R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ and this leads to Theorem 0.1. The restrictions on $V_{\mathbb{F}}$ in 0.2 arise because in these cases $R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ is not formally smooth over $\mathscr{O}$, and we know of no way to check that every component of $R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}[1/p]$ contains a crystalline point.

The result 0.2 is a local analogue of a theorem of Gouvêa-Mazur [**10**], extended by Böckle [**5**] which says that for a two dimensional $\mathbb{F}$-representation of the absolute Galois group of $\mathbb{Q}$, the generic fibre of the universal deformation space has a Zariski dense set of points corresponding to cusp forms on $\Gamma_1(N)$ (of various weights) where N is a suitable integer not divisible by p. The original argument of Gouvêa-Mazur uses the *eigencurve* [**7**], which is a kind of p-adic interpolation of these cusp forms. In particular, one can interpolate the global Galois representations attached to cusp forms into a family of Galois representations over the eigencurve.

In [**12**], we showed that the Galois representation attached to a point of the eigencurve admits at least one crystalline period. This local property (up to twist) was later dubbed *trianguline* by Colmez [**8**]. One of the results of [**12**] shows that 2-dimensional representations of $G_{\mathbb{Q}_p}$ with a crystalline period can be interpolated into a p-adic analytic space X_{fs}, which is a kind of local analogue of the eigencurve. Using it, one can imitate the arguments of Gouvêa-Mazur for local Galois representations, and show a statement about density of crystalline representations. This has also been carried out by Colmez, using his theory of Vector Spaces [**8**].

Finally let us mention that Paskunas [**18**] has shown that, when $\bar{\pi}$ is supersingular (that is, $V_{\mathbb{F}}$ is absolutely irreducible), then the surjection $R_{\bar{\pi},\psi} \to R_{V_{\mathbb{F}}}^{\psi\chi_{\mathrm{cyc}}}$ is an isomorphism. To prove this he shows directly that the dimension of the tangent space of the left hand side is at most 3, which is the dimension of the tangent space of the right hand side. (3) As a consequence one sees that, in this case, if π_A is a deformation of $\bar{\pi}$ with central character ψ, then $\det \mathbf{V}(\pi_A) = \psi\chi_{\mathrm{cyc}}$. Paskunas has also pointed out that this formula does not hold if $V_{\mathbb{F}}$ is unipotent.

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(3) Of course when $p = 3$ we continue to exclude the case $\begin{pmatrix} \omega_2^2 & 0 \\ 0 & \omega_2^6 \end{pmatrix} \otimes \chi$.

1. Density of crystalline representations

1.1. — Let $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. We will write $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, and we denote by $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character. As in the introduction, we denote by ω the mod p cyclotomic character, by $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ the inertia subgroup, and by ω_2 a fundamental character of level 2 of $I_{\mathbb{Q}_p}$.

Let $E/\mathbb{Q}_p$ be a finite extension. We will consider pairs (V,λ) consisting of a continuous representation of $G_{\mathbb{Q}_p}$ a two dimensional E-vector space V and $\lambda \in E^\times$ such that

1. $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V,V) = E$.
2. V is crystalline and the action of φ on
$$D_{\mathrm{cris}}(V^*) = \mathrm{Hom}_{E[G_{\mathbb{Q}_p}]}(V, B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)$$
has eigenvalues λ, λ' with $\lambda' \neq \lambda, p^{\pm 1}\lambda$.
3. V has Hodge-Tate weights $0, k$ with k a positive integer.

A pair (V,λ) satisfying the above condition will be called a *benign pair*. If V is a continuous representation of $G_{\mathbb{Q}_p}$ on a 2-dimensional E-vector space, then we say that V is benign if there exists a finite extension E'/E and $\lambda \in E'$ such that $(V \otimes_E E', \lambda)$ is benign. In particular, the condition (1) implies that V admits a universal deformation ring R_V.

Fix an E-basis of V. We denote by $R_V^\square$ the universal framed deformation ring of V. That is, if $\mathfrak{AR}_E$ denotes the category of Artinian local E-algebras with residue field E, then $R_V^\square$ represents the functor which to B in $\mathfrak{AR}_E$ assigns the set of isomorphism classes of pairs (V_B, β), where V_B is a deformation of V_E to B and β is a B-basis of V_B lifting the chosen basis of V_E. Recall that there is a natural map $R_V \to R_V^\square$ which is easily seen to be formally smooth of relative dimension 3.

Let (V,λ) be a benign pair. We denote by $D_V^{h,\varphi}$ the functor on $\mathfrak{AR}_E$ which assigns to B the set of isomorphism classes of deformations V_B of V_E to B such that, if
$$h : V \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} E)^{\varphi=\lambda}$$
is any non-zero E-linear, $G_{\mathbb{Q}_p}$-equivariant map, then h lifts to a map
$$\tilde{h} : V_B \to (B^+_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} B)^{\varphi=\tilde{\lambda}}$$
where $\tilde{\lambda} \in B^\times$ lifts λ. Note that (2) above implies that the set of maps h forms a torsor under $E^\times$. If V_B is in $D_V^{h,\varphi}(B)$, then the map $\tilde{h}$ is determined up to a unit in $B^\times$ and $\tilde{\lambda}$ is uniquely determined by λ [**12**, 8.12].

Let $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ denote the inertia subgroup. We have the following

Proposition 1.1. — *The functor $D_V^{h,\varphi}$ is pro-represented by a quotient $R_V^{h,\varphi}$ of R_V, which is formally smooth over E of dimension 3. The composite*
$$I_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p} \to R_V^{h,\varphi\times}$$
given by the determinant of the universal deformation, does not factor through $E^\times$.

Proof. — This is [**12**, 10.2]. □

Proposition 1.2. — *Let $h \in D_{\mathrm{cris}}(V^*)^{\varphi=\lambda}$ and $h' \in D_{\mathrm{cris}}(V^*)^{\varphi=\lambda'}$ be non-zero. Then the closed subschemes* $\operatorname{Spec} R_V^{h,\varphi}$ *and* $\operatorname{Spec} R_V^{h',\varphi}$ *of* $\operatorname{Spec} R_V$ *are distinct. More precisely,* $\operatorname{Spec} R_V^{h,\varphi} \otimes_{R_V} R_V^{h',\varphi}$ *is a smooth subscheme of* $\operatorname{Spec} R_V$ *of dimension* 2.

Proof. — Let B be in $\mathfrak{AR}_E$ and V_B in $D_V(B)$. It is not hard to check that V_B is in $D_V^{h,\varphi}(B)$ and $D_V^{h',\varphi}(B)$ if and only if V_B is crystalline. For example, use [**12**, 8.9]. The proposition now follows from [**13**, Thm. 3.3.8], which shows that the preimage of $\operatorname{Spec} R_V^{h,\varphi} \otimes_{R_V} R_V^{h',\varphi}$ in $\operatorname{Spec} R_V^{\square}$ is formally smooth and 5-dimensional. □

1.2. — Let $\mathbb{F}$ be a finite field of characteristic p and $V_{\mathbb{F}}$ a two dimensional $\mathbb{F}$-vector space equipped with a continuous action of $G_{\mathbb{Q}_p}$. We fix an $\mathbb{F}$-basis of $V_{\mathbb{F}}$.

Let $\mathfrak{AR}_{W(\mathbb{F})}$ denote the category of local Artinian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. We denote by $R_{V_{\mathbb{F}}}^{\square}$ the universal framed deformation ring of $V_{\mathbb{F}}$. That is, $R_{V_{\mathbb{F}}}^{\square}$ is the complete local $W(\mathbb{F})$-algebra which prorepresents the functor assigning to A in $\mathfrak{AR}_{W(\mathbb{F})}$ the set of isomorphism classes of pairs (V_A, β), where V_A is a deformation of the $G_{\mathbb{Q}_p}$-representation $V_{\mathbb{F}}$ to A, and β is an A-basis of V_A lifting the chosen $\mathbb{F}$-basis of $V_{\mathbb{F}}$. We set $Z = \operatorname{Spec} R_{V_{\mathbb{F}}}^{\square}[1/p]$.

If $E/W(\mathbb{F})[1/p]$ is a finite extension and $x : R_{V_{\mathbb{F}}}^{\square} \to E$ an E-valued point, then x gives rise to a two dimensional E-representation of $G_{\mathbb{Q}_p}$, equipped with an E-basis. Let $\hat{R}_{V_{\mathbb{F}},x}^{\square}$ denote the completion of $R_{V_{\mathbb{F}}}^{\square}[1/p]$ at the maximal ideal generated by the kernel of x. Then $\hat{R}_{V_{\mathbb{F}},x}^{\square} \otimes_{\kappa(x)} E$ is canonically isomorphic to $R_{V_x}^{\square}$, the framed deformation ring of V_x [**15**, 2.3.5]. Here $\kappa(x)$ denotes the residue field of x.

We call x benign if V_x is benign. We say that $(x, \lambda) \in (Z \times \mathbb{G}_m)(E)$ is benign if (V_x, λ) is a benign pair.

Let S denote the universal deformation ring of $\det V_{\mathbb{F}}$, thought of as a representation of the inertia subgroup of the maximal abelian quotient of $G_{\mathbb{Q}_p}$. Then S is formally smooth over $W(\mathbb{F})$ of relative dimension 1 if $p > 2$ and isomorphic to $W(\mathbb{F})[\![Y, Z]\!]/((1+Z)^2 - 1)$ if $p = 2$. We have an obvious map

$$\operatorname{Spec} R_{V_{\mathbb{F}}}^{\square} \xrightarrow{V_A \mapsto \det V_A|_{I_{\mathbb{Q}_p}}} \operatorname{Spec} S.$$

In the following we will use the construction of the p-adic analytic space attached to $\operatorname{Spec} R[1/p]$ where R is a complete local $W(\mathbb{F})$-algebra with finite residue field [**11**, § 7]. In particular, we write $\mathscr{W} = \operatorname{Spec} S[1/p]$ and we denote by $\mathscr{W}^{\mathrm{an}}$ the associated p-adic analytic space. (4)

Proposition 1.3. — *There exists a reduced, Zariski closed, analytic subspace* $X_{fs} \subset (Z \times \mathbb{G}_m)^{\mathrm{an}}$ *with the following properties.*

(4) $\mathscr{W}$ and $\mathscr{W}^{\mathrm{an}}$ are what is usually referred to as "weight space".