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DEFORMATION RINGS FOR SOME MOD 3 GALOIS REPRESENTATIONS OF THE ABSOLUTE GALOIS GROUP OF $\mathbf{Q}_3$

by

Gebhard Böckle

Abstract. — In this note we compute the (uni)versal deformation of two types of mod 3 Galois representations $\bar{\rho} : \mathrm{GL}(\overline{\mathbb{Q}_3}/\mathbb{Q}_3) \to \mathrm{GL}_2(\overline{\mathbb{F}_3})$. In the cases considered the (uni)versal ring is obstructed. Our main result is that the ring is still an integral domain. The result has consequences for the p-adic local Langlands correspondence: By work of Colmez and Kisin it allows one to deduce that benign crystalline points are Zariski dense in the universal space for $p = 3$. Thus the p-adic local Langlands correspondence [**4**] as well as the result [**6**] have no longer any exceptional cases for $p = 3$.

Résumé **(Anneaux de déformation pour certaines représentations galoisiennes mod 3 du groupe de Galois absolu de $\mathbf{Q}_3$)**

Dans cette note, nous calculons la déformation (uni)verselle de deux types de représentations galoisiennes $\bar{\rho} : \mathrm{GL}(\overline{\mathbb{Q}_3}/\mathbb{Q}_3) \to \mathrm{GL}_2(\overline{\mathbb{F}_3})$. Dans les cas que nous considérons, l'anneau (uni)versel est obstrué. Notre résultat principal énonce que l'anneau reste intègre. Ce résultat a des conséquences pour la correspondance de Langlands p-adique locale : en utilisant les travaux de Colmez et de Kisin, il nous permet de déduire que les points crystallins bénins sont denses pour la topologie de Zariski, dans l'espace universel pour $p = 3$. Ainsi la correspondance de Langlands p-adique locale [**4**] ainsi que le résultat de [**6**] n'ont plus de cas exceptionnels pour $p = 3$.

1. Introduction

Let p be a prime, let $\mathbb{Q}_p$ denote the completion of the field of rational numbers $\mathbb{Q}$ under the p-adic norm and let $K \supset \mathbb{Q}_p$ be a finite extension field. For q a power of p denote by $\mathbb{F}_q$ the field of q elements and by $\mathbb{Z}_q$ the ring of Witt vectors of $\mathbb{F}_q$, so that $\mathbb{Z}_q$ is the complete discrete valuation ring of characteristic zero with uniformizer p and residue field $\mathbb{F}_q$. Consider a continuous representation

$$\bar{\rho} \colon G_K \to \mathrm{GL}_2(\mathbb{F}_q)$$

of the absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K.

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To $\bar{\rho}$ we apply the deformation theory developed by Mazur [**9**]: Let CNL_q denote the category of complete noetherian local $\mathbb{Z}_q$-algebras R with residue field $\mathbb{F}_q$. The algebra structure yields a canonical surjective homomorphism $\pi_R\colon R \to \mathbb{F}_q$ of $\mathbb{Z}_q$-algebras. Its kernel is the maximal ideal of R which we denote by $\mathfrak{m}_R$. For $R \in \mathrm{CNL}_q$, a *lift* of $\bar{\rho}$ to R is a continuous representation $\rho\colon G_K \to \mathrm{GL}_2(R)$ such that $\bar{\rho} = \mathrm{GL}_2(\pi_R) \circ \rho$. A *deformation* is a strict equivalence class of lifts where two lifts are strictly equivalent if they are in the same conjugacy class under conjugation by matrices in $\Gamma(R) := \mathrm{Ker}(\mathrm{GL}_2(\pi_R) : \mathrm{GL}_2(R) \to \mathrm{GL}_2(\mathbb{F}_q)) \subset \mathrm{GL}_2(R)$. Following Mazur one considers the functor which to any R in CNL_q associates the set of all deformations of $\bar{\rho}$ to R.

By [**9**] this functor always has a versal hull. The versal hull is a strict equivalence class of a lift $\rho_v\colon G_K \to \mathrm{GL}_2(R_v)$ of $\bar{\rho}$ which is characterized (up to isomorphism) by the following two properties: (a) any deformation to a ring R is obtained as the composite of ρ_v with a $\mathbb{Z}_q$-algebra homomorphism $R_v \to R$ in CNL_q; (b) the composition of ρ_v with the canonical surjection $R_v \twoheadrightarrow R_v/(\mathfrak{m}^2_{R_v}, p)$ is universal for deformations to $\mathbb{F}_q[\varepsilon]/(\varepsilon^2)$. The versal hull is universal if $\dim_{\mathbb{F}_q} H^0(G_K, \mathrm{ad}) = 1$; here ad denotes the adjoint representation of G_K on the set of 2×2 matrices $M_2(\mathbb{F}_q)$ over $\mathbb{F}_q$, i.e., the composite of $\bar{\rho}$ with the conjugation action of $\mathrm{GL}_2(\mathbb{F}_q)$ on $M_2(\mathbb{F}_q)$.

In this note we shall explicitly compute the versal deformation rings for two (types of) $\bar{\rho}$ in the case where $K = \mathbb{Q}_3$, and so from now on we specialize p to 3. For every $n \in \mathbb{N}$ we fix a primitive n-th root of unity $\zeta_n \in \overline{\mathbb{Q}_3}$. We define $\chi_3\colon G_{\mathbb{Q}_3} \to \mathbb{Z}/(3)^* \cong \mathbb{F}_3^*$ as the mod 3 cyclotomic character, so that $g\zeta_3 = \zeta_3^{\chi_3(g)}$ for $g \in G_{\mathbb{Q}_3}$. We also define characters $\omega_i\colon G_{\mathbb{Q}_{3^i}} \to \mathbb{F}_{3^i}^*$, $i = 1, 2$, by

$$\sigma \mapsto \omega_i(\sigma) \equiv \frac{\sigma(\sqrt[3^i-1]{3})}{\sqrt[3^i-1]{3}} \pmod{3\,\mathbb{Z}_{3^i}};$$

the fraction on the right is a primitive $(3^i - 1)$-th root of unity in $\mathbb{Z}_{3^i}$. The characters ω_i are totally and tamely ramified.

We shall study the following two (types of) residual mod 3 Galois representations $\bar{\rho}_i\colon \mathrm{Gal}(\overline{\mathbb{Q}_3}/\mathbb{Q}_3) \to \mathrm{GL}_2(\overline{\mathbb{F}_3})$: By $\bar{\rho}_1$ we denote a representation which is an extension of the trivial character by χ_3, so that

$$\bar{\rho}_1\colon G_{\mathbb{Q}_3} \to \mathrm{GL}_2(\mathbb{F}_q) : \sigma \mapsto \begin{pmatrix} \chi_3(\sigma) & \beta(\sigma) \\ 0 & 1 \end{pmatrix}$$

for some power q of 3; here $\sigma \mapsto \beta(\sigma)$ is a continuous 1-cocycle and the set of $\bar{\rho}_1$ up to isomorphism is in bijection with $H^1_{\mathrm{cont}}(G_{\mathbb{Q}_3}, \mathbb{F}_q^{\chi_3})$. If $0 = [\beta] \in H^1_{\mathrm{cont}}(G_{\mathbb{Q}_3}, \mathbb{F}_q^{\chi_3})$ we choose $\beta = 0$. From local Tate duality and the local Euler-Poincaré formula, cf. [**10**, §3], one deduces

$$\begin{aligned}&\dim_{\mathbb{F}_q} H^1_{\mathrm{cont}}(G_{\mathbb{Q}_3}, \mathbb{F}_q^{\chi_3}) = \\ &\quad \dim \mathbb{F}_q^{\chi_3} + \dim H^0_{\mathrm{cont}}(G_{\mathbb{Q}_3}, \mathbb{F}_q^{\chi_3}) + \dim H^0_{\mathrm{cont}}(G_{\mathbb{Q}_3}, (\mathbb{F}_q^{\chi_3})^*(\chi_3)) = 1 + 0 + 1 = 2.\end{aligned}$$

By $\bar{\rho}_2$ we denote the induced representation

$$\bar{\rho}_2 := \mathrm{Ind}_{G_{\mathbb{Q}_9}}^{G_{\mathbb{Q}_3}} \omega_2^2 \colon G_{\mathbb{Q}_3} \to \mathrm{GL}_2(\mathbb{F}_3);$$

we remark that the image of $\bar{\rho}_2$ is a dihedral group of order 8 of which it is known that its irreducible degree 2 representation on $\overline{\mathbb{F}_3}$ is defined over $\mathbb{F}_3$. To have a uniform notation for the coefficient fields for both $\bar{\rho}_i$, we take $q = 3$ for the representation $\bar{\rho}_2$.

Let $\rho_i : G_{\mathbb{Q}_3} \to \mathrm{GL}_2(R_i)$ denote the versal hull of $\bar{\rho}_i$. One easily verifies that it is universal if either $i = 2$ or if $i = 1$ and $[\beta] \neq 0$—note that $\dim H^0(G_{\mathbb{Q}_3}, \mathrm{ad}) = 2$ if $i = 1$ and $[\beta] = 0$. The main result of this article is an explicit computation of R_i which leads to the following result:

Theorem 1.1. — *The ring R_i is an integral domain. Moreover R_i is a local complete intersection, flat over $\mathbb{Z}_q$ and of relative dimension $4 + \dim H^0(G_{\mathbb{Q}_3}, \mathrm{ad})$.*

The proof follows closely that of the main result [**1**, Theorem 2.6]. The new assertion made, in comparison with [**1**], is that the rings R_i for the two cases at hand are integral domains. This implies that the $\mathrm{Spec}(R_i[1/3])$ are reduced and irreducible.

By [**6**, Cor. 1.3.6], the $\bar{\rho}_i$ considered here are precisely those 2-dimensional representations of $G_{\mathbb{Q}_3}$ over a finite extension of $\mathbb{F}_3$ for which Mazur's deformation functor is obstructed, i.e., for which $H^2(G_{\mathbb{Q}_3}, \mathrm{ad}) \neq 0$. Thus Theorem 1.1 holds for the (uni)versal deformation rings of all such residual representations. Moreover one can easily adapt the (methods of the) present article to study deformation functors for deformations having a fixed determinant ψ as in [**6**]. The corresponding (uni)versal deformation ring satisfies all assertions of Theorem 1.1 except that its relative dimension is $3 + \dim H^0(G_{\mathbb{Q}_3}, \mathrm{ad}^0)$.

Since $\mathrm{Spec}\, R_i[1/3]$ is irreducible, [**3**, § 6] or [**6**, Cor. 1.3.4] imply that trianguline or benign crystalline points are Zarisiki dense in it—as well as the analogous result for deformations with a fixed determinant (note that [**6**, Cor. 2.3.7] only needs cases of the present note in which $\dim H^0(G_{\mathbb{Q}_3}, \mathrm{ad}^0) = 0$, i.e., those in which R_i is universal). By this, the p-adic local Langlands correspondence [**4**, in part. Thme. II.3.3] and the result [**6**, Thms. 0.1 and 0.3] have no longer any exceptional cases for $p = 3$.

We now survey the present article. In Section 2 Mazur's deformation functor for the $\bar{\rho}_i$ considered here is identified as a functor describing sets of equivariant homomorphisms from the pro-3 completion P of the absolute Galois group of an extension of $\mathbb{Q}_3$ determined by $\bar{\rho}_i$ to a pro-3 Sylow subgroup of $\mathrm{GL}_2(R)$—the idea to consider functors of equivariant homomorphisms goes back to Boston, e.g. [**2**]. The group P is a Demuškin group which carries an action of $\mathrm{Im}(\bar{\rho})$ modulo its normal 3-Sylow subgroup U. In Section 3, we recall the main results on such groups.

Compared to the results in [**1**] there are two improvements. In Section 2 the Demuškin group P arises from an extension of $\mathbb{Q}_3$ that is possibly of a smaller degree than in [**1**] or [**2**]. This facilitates the computations related to $\bar{\rho}_2$. In Section 3 we are able to give an explicit presentation of P in terms of topological generators and one relation r where on the generators and thus also on r the action of $\mathrm{Im}(\bar{\rho})/U$ is also given explicitly! For $\bar{\rho}_1$ this was indicated in [**1**, Example 3.7]. For $\bar{\rho}_2$ this is new and rather simple—but it was not noticed in [**1**]. The explicit form of r will in Sections 4 and 5 allow the explicit computation of the versal deformations ρ_i. The Rings R_i

are given as the quotient of a power series ring over $\mathbb{Z}_q$ by ideals whose generators can in principle be given explicitly. However the actual generators we find are too complicated to write down. Instead, using a computer algebra package, we can give truncated power series to sufficient high precision to prove Theorem 1.1.

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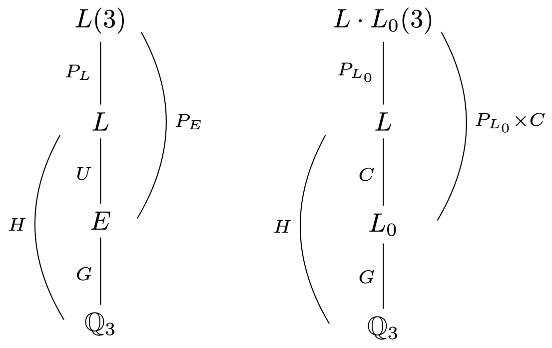
2. A functor of equivariant homomorphisms

For a field k let k^{sep} denote a fixed separable closure. We define P_k as the pro-3 completion of $G_k := \text{Gal}(k^{\text{sep}}/k)$. This is a quotient of G_k by a closed normal subgroup. The fixed field of this subgroup inside k^{sep} we denote by $k(3)$.

We introduce various extension fields of $\mathbb{Q}_3$ inside $\overline{\mathbb{Q}_3}$ and Galois groups—they depend on $\bar\rho_i$ but we omit this dependency in the notation. The *splitting field* of $\bar\rho$ is $L := G_{\mathbb{Q}_3}^{\text{Ker}(\bar\rho_i)}$. The group $H := \text{Gal}(L/\mathbb{Q}_3)$ has a unique 3-Sylow subgroup denoted U—it is trivial for $\bar\rho_2$. For $\bar\rho_1$, the fixed field L^U is $E := \mathbb{Q}_3(\zeta_3)$ and we write $G := \text{Gal}(E/\mathbb{Q}_3)$. Since U is a 3-group one has $E(3) = L(3)$. For $\bar\rho_2$, we define $L_0 := G_{\mathbb{Q}_3}^{\text{Ker(ad)}}$ as the splitting field of ad and we set $C := \text{Gal}(L/L_0)$ and $G := \text{Gal}(L_0/\mathbb{Q}_3)$. For the convenience of the reader, we display the situations for both $\bar\rho_i$ in the following diagrams:



for $\bar\rho_1$ for $\bar\rho_2$

In the diagram for $\bar\rho_1$ the group G is isomorphic to a cyclic group of order 2, say $G = \{1, \sigma\}$. The group U is of order 1, 3 or 9 as can be deduced from $\dim_{\mathbb{F}_3} H^1(G_{\mathbb{Q}_3}, \mathbb{F}_3^{\chi_3}) = 2$. If U is non-trivial, we denote by $u \in U$ a non-trivial element. By conjugating $\bar\rho_1$ suitable, we may then assume that $\bar\rho_1(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In [**2**, §2] a profinite version of the Lemma of Schur-Zassenhaus is stated. It implies that $\text{Gal}(L(3)/\mathbb{Q}_3)$ is isomorphic to a semi-direct product $P_E \rtimes G$. We thus fix a lift of the generator σ of $\text{Gal}(E/\mathbb{Q}_3)$

to $\mathrm{Gal}(L(3)/\mathbb{Q}_3)$ of order 2. By the quoted Schur-Zassenhaus lemma any two such lifts are conjugate by an inner automorphism.

In the diagram for $\bar{\rho}_2$ the group H is a dihedral group of order 8 and its quotient G is a Klein 4-group. Because P_{L_0} is a pro-3 group and the index $[L:L_0]$ is 2, one has $L \cap L_0(3) = L_0$ and thus $\mathrm{Gal}(LL_0(3)/L_0)$ is isomorphic to the product $P_{L_0} \times C$. Again by the profinite Schur-Zassenhaus lemma, we have $\mathrm{Gal}(LL_0(3)/\mathbb{Q}_3) \cong P_{L_0} \rtimes H$ where the subgroup $C \subset H$ acts trivially on P_{L_0}. As above we fix a splitting of $\mathrm{Gal}(LL_0(3)/\mathbb{Q}_3) \to H$ and note that any two such differ by an inner automorphism.

Define $U_2(\mathbb{F}_q) \subset \mathrm{GL}_2(\mathbb{F}_q)$ as the subgroup of upper triangular matrices with 1's on the diagonal and define for any $R \in \mathrm{CNL}_q$ the group $\tilde{\Gamma}(R)$ as $\mathrm{GL}_2(\pi_R)^{-1}(U_2(\mathbb{F}_q))$, so that:

$$\Gamma(R) = \mathrm{GL}_2(\pi_R)^{-1}(\{1\}) \subset \tilde{\Gamma}(R) \subset \mathrm{GL}_2(R).$$

The groups $\Gamma(R)$ and $\tilde{\Gamma}(R)$ are pro-3 groups. It follows from [**2**, §6.9] that any lift $\rho\colon G_{\mathbb{Q}_3} \to \mathrm{GL}_2(R)$ of $\bar{\rho}_i$ contains $\mathrm{Gal}(\overline{\mathbb{Q}_3}/L(3))$ in its kernel. But for $\bar{\rho}_2$ slightly more is true.

Lemma 2.1. — *Any lift* $\rho : G_{\mathbb{Q}_3} \to \mathrm{GL}_2(R)$ *of* $\bar{\rho}_2$ *contains* $\mathrm{Gal}(\overline{\mathbb{Q}_3}/LL_0(3))$ *in its kernel.*

Proof. — The image of C under $\bar{\rho}_2$ is the set $\{\pm 1_2\}$ where 1_2 is the identity matrix in $\mathrm{GL}_2(\mathbb{F}_q)$. If we denote by 1_2 the same matrix in $\mathrm{GL}_2(R)$ it follows that

$$\mathrm{GL}_2(\pi_R)^{-1}(\{\pm 1_2\}) \cong \Gamma(R) \times \{\pm 1_2\} \subset \mathrm{GL}_2(R).$$

By the profinite Schur-Zassenhaus lemma any element of order 2 in $\mathrm{GL}_2(\pi_R)^{-1}(\{\pm 1_2\})$ is conjugate to -1_2 and hence equal to -1_2 since this element is central. By the same lemma $\rho(\mathrm{Gal}(\overline{\mathbb{Q}_3}/L_0)) \subset \mathrm{GL}_2(\pi_R)^{-1}(\{\pm 1_2\})$ is a semidirect product of a group of order 2 and a pro-p group. Up to strict equivalence we may assume that the group of order 2 is generated by the central element $-1_2 \in \mathrm{GL}_2(R)$. Hence $\rho(\mathrm{Gal}(\overline{\mathbb{Q}_3}/L_0))$ is a product of a pro-3 group with $\{\pm 1_2\}$. In particular, the pro-3 group is the Galois group of a Galois extension of L_0, and thus of a subextension of $L_0(3)$. □

We now define functors $\mathrm{EH}_i\colon \mathrm{CNL}_q \to \mathbf{Sets}$ of equivariant homomorphisms corresponding to the $\bar{\rho}_i$ as follows: To $R \in \mathrm{CNL}_q$ we associate

$$\mathrm{EH}_1(R) := \left\{\alpha \in \mathrm{Hom}_{G,\mathrm{cont}}\left(P_E, \tilde{\Gamma}(R)\right) \,\middle|\, \alpha \bmod \mathfrak{m}_R = \bar{\rho}_1|_{G_E} \text{ and } \alpha(u) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \text{ if } U \neq 0\right\}$$

if $i=1$, and we associate $\mathrm{EH}_2(R) := \mathrm{Hom}_{H,\mathrm{cont}}(P_{L_0}, \Gamma(R))$ if $i=2$. Again by Schur-Zassenhaus, if $i=1$ we fix a homomorphism $\lambda_1\colon \mathrm{Gal}(E/\mathbb{Q}_3) \to \mathrm{GL}_2(\mathbb{Z}_q)$ whose mod 3 reduction is the composite of $\bar{\rho}_1$ with a splitting of $\mathrm{Gal}(L/\mathbb{Q}_3) \to \mathrm{Gal}(E/\mathbb{Q}_3)$, and if $i=2$ a homomorphism $\lambda_2\colon \mathrm{Gal}(L/\mathbb{Q}_3) \to \mathrm{GL}_2(\mathbb{Z}_3)$ which is a lift of $\bar{\rho}_2$. The following is a variant of [**1**, Prop. 2.3]; its proof is left to the reader who may consult [**2**, §6,9].

Proposition 2.2. — *The functors* EH_i *are representable. Let* $(\tilde{R}_i, \tilde{\alpha}_i)$ *denote a universal pair and define the continuous representation* $\tilde{\rho}_i\colon G_{\mathbb{Q}_3} \to \mathrm{GL}_2(\tilde{R}_3)$ *by*

$$\tilde{\rho}_i((h,g)) := \alpha_i(h)\lambda_i(g)$$