Astérisque
323, 2009, p. 61–81

THE SPACE OF GENERALIZED FORMAL POWER SERIES SOLUTIONS OF AN ORDINARY DIFFERENTIAL EQUATION

by

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Abstract. — We prove that the set of truncations of generalized power series solutions of an ordinary differential equations is contained in a semi-algebraic set of dimension bounded by twice the order of the differential equation.

Résumé **(L'espace des séries formelles généralisées qui sont solution d'une équation différentielle ordinaire)**
Nous montrons que l'ensemble des troncations de séries généralisées qui sont solutions d'une équation différentielle ordinaire est contenu dans un ensemble semi-algébrique dont la dimension est bornée par le double de l'ordre de l'équation différentielle.

1. Introduction

Consider a polynomial differential equation $F(\partial_0(y),\ldots,\partial_n(y)) = 0$, where $F(y_0,\ldots,y_n)$ is a polynomial in the variables $y_0,\ldots,y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$ (polynomials with real exponents). We are interested in series solutions of $(F = 0)$ of the form $\sum_{i=1}^{\infty} c_i\, x^{\mu_i}$, where $c_i \in \mathbb{C}$ and $\mu_i \in \mathbb{R}$ with $\mu_1 < \mu_2 \cdots$ (so called *generalized power series*). D.Y. Grigor'ev and M. Singer describe in [**5**] a parametric version of the Newton polygon process applied to F, which for each integer k, gives rise to a semi-algebraic subset $\mathrm{NIC}_k^{\star}(F) \subseteq \mathbb{R}^{3k}$ so that the space of truncations of length k of generalized power series solution of $(F = 0)$ is included in $\mathrm{NIC}_k^{\star}(F)$. The main contribution of this paper is to prove that the dimension of this semi-algebraic set is bounded by $2n$. More precisely, its *adapted dimension* (see subsection 3.2) is bounded by n. The adapted dimension is a proper measure of the *number of free parameters* (real or complex, coefficient or exponent) which have been introduced

2010 Mathematics Subject Classification. — 37F75.
Key words and phrases. — ODE, generalized series.

The authors wish to thank the referee for many helpful suggestions.

along the Newton polygon process in a parametric family of power series solution of a differential equation.

Briot and Bouquet [**1**] in 1856 use the Newton polygon for studying first order and first degree ordinary differential equations and Fine [**4**] in 1889 gives a description of the method for ordinary differential equation of arbitrary order. In section 2 we present a brief introduction to its classical version. In section 4 we introduce the notion of parametric Newton polygon: specifically, we define it and give some technical results about *parametric polynomials* which will be used in the proof of the main theorem.

In section 3 we state the main theorem and give a straightforward proof for the case $k = 1$. The general case is dealt with in section 5.

2. Newton polygon of an ODE

A *well-ordered* series with complex coefficients and real exponents is a series $\phi(x) = \sum_{\alpha \in S} c_\alpha\, x^\alpha$, where $c_\alpha \in \mathbb{C}$, and S is a well ordered subset of $\mathbb{R}$. If there exist a finitely generated semi-group Γ of $\mathbb{R}_{\geq 0}$ and $\gamma \in \mathbb{R}$, such that, $S \subseteq \gamma + \Gamma$, then we say that $\phi(x)$ is a *grid-based* series (this terminology comes from [**6**].) Let $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ be the sets of well-ordered series and of grid-based series, respectively. We denote $\mathbb{C}[x^{\mathbb{R}}]$ the subring of series in $\mathbb{C}((x))^g$ with finite support (*polynomials*, so to speak). It is well-know (see [**7**], for example), that both $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ are actually fields. Both are differential rings with the usual inner operations and the differential operator $\partial = x\frac{d}{dx}$:

$$\partial\left(\sum c_\alpha\, x^\alpha\right) = \sum \alpha\, c_\alpha\, x^\alpha.$$

Denote by ∂_0 the identity operator and for positive integer i, $\partial_i = \partial \circ \partial_{i-1}$.

Let $F(y_0, \dots, y_n)$ be a polynomial in the variables $y_0, \dots, y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$. The differential equation

$$F\big(\partial_0(y), \partial_1(y), \dots, \partial_n(y)\big) = 0$$

will be denoted by $F(y) = 0$. Notice that any polynomial ordinary differential equation can be rewritten in this form.

We are interested in solutions of $F(y) = 0$ in the field $\mathbb{C}((x))^w$. By virtue of [**2, 5, 6**], all of them are actually in $\mathbb{C}((x))^g$.

Write F in a uniquely, using the standard multiindex notation $y^\rho = y_0^{\rho_0} \cdots y_n^{\rho_n}$ (where $\rho = (\rho_0, \dots, \rho_n)$) as

$$F = \sum_{\alpha,\rho} A_{\alpha,\rho}\, x^\alpha\, y^\rho, \text{ with } A_{\alpha,\rho} \in \mathbb{C},$$

where α and ρ run over finite subsets of $\mathbb{R}$ and $\mathbb{N}^{n+1}$ respectively. The *cloud of points* of F is the set

$$\mathscr{P}(F) = \{(\alpha, |\rho|) : A_{\alpha,\rho} \neq 0\},$$

where $|\rho| = \rho_0 + \cdots + \rho_n$. The Newton polygon $\mathscr{N}(F)$ of F is the convex hull of

$$\bigcup_{P \in \mathscr{P}(F)} (P + \{(a,0) \mid a \geq 0\}) \ .$$

Notice that $\mathscr{N}(F)$ has a finite number of vertices, all of whose ordinates are non-negative integers.

Given a line $L \subseteq \mathbb{R}^2$ with slope $-1/\mu$, we say that μ is the *inclination* of L. Let $\mu \in \mathbb{R}$, we denote $L(F;\mu)$ the supporting line of $\mathscr{N}(F)$ with inclination μ (i.e. the only line L with inclination μ such that $\mathscr{N}(F)$ is contained in the right closed half-plane defined by L and $L \cap \mathscr{N}(F) \neq \varnothing$). More precisely, $L(F;\mu)$ is the set of points (a,b) in $\mathbb{R}^2$ such that $a + \mu b = \nu(F;\mu)$, where $\nu(F;\mu) = \min\{\alpha + \mu\,|\rho|\,; A_{\alpha,\rho} \neq 0\}$.

For any $\mu \in \mathbb{R}$, define the polynomial

(1) $$\Phi_{(F;\mu)}(\mathfrak{c}) = \sum_{(\alpha,|\rho|) \in L(F;\mu)} A_{\alpha,\rho}\, \mu^{w(\rho)}\, \mathfrak{c}^{|\rho|} \in \mathbb{C}[\mathfrak{c}],$$

where $w(\rho) = \rho_1 + 2\rho_2 + \cdots + n\rho_n$. The *Newton polygon data* of F will be the set of vertices $v_0, \ldots, v_t$ (ordered with decreasing ordinate), the sides $[v_i, v_{i+1}]$, $0 \leq i < t$, the *indicial polynomials* associated to each vertex v:

(2) $$\Psi_{(F;v)}(\mathfrak{m}) = \sum_{(\alpha,|\rho|) = v} A_{\alpha,\rho}\, \mathfrak{m}^{w(\rho)} \in \mathbb{C}[\mathfrak{m}].$$

and the *characteristic polynomials* associated to each side $[v_i, v_{i+1}]$:

$$\Phi_{(F;[v_i,v_{i+1}])}(\mathfrak{c}) = \Phi_{(F;\mu_{[v_i,v_{i+1}]})}(\mathfrak{c}),$$

where $\mu_{[v_i,v_{i+1}]}$ is the inclination of side $[v_i, v_{i+1}]$.

2.1. Necessary Initial Conditions. — Given a well-ordered formal power series $y(x) = \sum_{\alpha \in S} c_\alpha\, x^\alpha$, its *order*, $\mathrm{ord}(y(x))$, is infinity if $y(x) = 0$ and $\min\{\alpha \in S \mid c_\alpha \neq 0\}$ otherwise.

Lemma 1. — *Let $y(x) = c\, x^\mu + \sum_{\alpha > \mu} c_\alpha\, x^\alpha \in \mathbb{C}((x))^w$ be a solution of the differential equation $F(y) = 0$. Then*

$$\Phi_{(F;\mu)}(c) = 0.$$

where c may be zero. In particular, if $y(x) = 0$ is a solution of $F(y) = 0$ then $\Phi_{(F;\mu)}(0) = 0$ for all μ.

Proof. — Developing F

$$\begin{aligned}&F(c\,x^{\mu}+\cdots)=\\&\sum_{\alpha,\rho}A_{\alpha,\rho}\;x^{\alpha}\,(c\,x^{\mu}+\cdots)^{\rho_0}(\mu c\,x^{\mu}+\cdots)^{\rho_1}\cdots(\mu^{n}c\,x^{\mu}+\cdots)^{\rho_n}=\\&\sum_{\alpha,\rho}\left\{A_{\alpha,\rho}\;c^{|\rho|}\;\mu^{w(\rho)}x^{\alpha+\mu|\rho|}+\cdots\right\}=\\&\left\{\sum_{\alpha+\mu\,|\rho|=\nu(F;\mu)}A_{\alpha,\rho}\;c^{|\rho|}\;\mu^{w(\rho)}\right\}x^{\nu(F;\mu)}+\cdots,\end{aligned}$$

where dots $\cdots$ stand for monomials of order greater than the exponent of x in the preceding term. The lemma follows from the fact that $\alpha+\mu\,|\rho|=\nu(F;\mu)$ if and only if $(\alpha,|\rho|)\in L(F;\mu)$. □

Notation 1. — *Let $\varphi\in\mathbb{C}((x))^{g}$ and $F(y_0,\dots,y_n)\in\mathbb{C}((x))^{g}[y_0,\dots,y_n]$, denote*

$$F(\varphi+y)=F(\varphi+y_0,\partial(\varphi)+y_1,\dots,\partial_n(\varphi)+y_n)\in\mathbb{C}((x))^{g}[y_0,\dots,y_n].$$

Definition 1. — *Given $F(y_0,\dots,y_n)$ and a positive integer k, define the set of* necessary k-initials conditions, $\mathrm{NIC}_k(F)$, *to be the subset of $(\mathbb{R}\times\mathbb{C})^{k}$ of the points $(\mu_1,c_1,\dots,\mu_k,c_k)\in(\mathbb{R}\times\mathbb{C})^{k}$ such that*

$$\mu_1<\cdots<\mu_k,\quad \textit{and}$$
$$\Phi_{(F_1;\mu_1)}(c_1)=0,\dots,\Phi_{(F_k;\mu_k)}(c_k)=0,$$

where $F_1(y)=F(y)$ and $F_{i+1}(y)=F_i(c_i\,x^{\mu_i}+y)$, for $1\le i<k$.

Define the $\mathrm{NIC}_k^{*}(F)=\mathrm{NIC}_k(F)\cap(\mathbb{R}\times\mathbb{C}^{*})^{k}$, *where $\mathbb{C}^{*}=\mathbb{C}\setminus\{0\}$.*

Corollary 1. — *If $y(x)=\sum_{i=1}^{k}c_i x^{\mu_i}+\sum_{\mu_k<\alpha}c_\alpha x^{\alpha}$ is a solution of $F(y)=0$ with $\mu_1<\cdots<\mu_k$, then*

$$(\mu_1,c_1,\dots,\mu_k,c_k)\in\mathrm{NIC}_k(F).$$

Corollary 2. — *Let $v_0,\dots,v_t$ be the vertices of $\mathscr{N}(F)$, ordered by decreasing ordinate. Let μ_i, $1\le i\le t$ be the inclination of the side $[v_{i-1},v_i]$. Set $\mu_0=-\infty$ and $\mu_{t+1}=+\infty$. The subset $\mathrm{NIC}_1(F)\subseteq(\mathbb{R}\times\mathbb{C})$ is semi-algebraic. Moreover, $\mathrm{NIC}_1^{*}(F)$ is the finite union of the semi-algebraic sets corresponding to the sides of the Newton polygon of F:*

$$\{(\mu,c)\in\mathbb{R}\times\mathbb{C}^{*}\,;\mu=\mu_i,\ \textit{and}\ \Phi_{(F;\mu_i)}(c)=0\},\quad 1\le i\le t,$$

and the semi-algebraic sets corresponding to the vertices:

$$\{(\mu,c)\in\mathbb{R}\times\mathbb{C}^{*}\,;\mu_i<\mu<\mu_{i+1},\ \textit{and}\ \Psi_{(F;v_i)}(\mu)=0\},\quad 0\le i\le t.$$

Proof. — Let $\mu\in\mathbb{R}$, $\mu_i<\mu<\mu_{i+1}$, for some $0\le i\le t$. As $L(F;\mu)\cap\mathscr{N}(F)=v_i$ and $\Phi_{(F;\mu)}(\mathfrak{c})=\mathfrak{c}^{h}\,\Psi_{(F;v_i)}(\mu)$, (where h is the ordinate of v_i) then, for $c\neq0$ and $\mu_i<\mu<\mu_{i+1}$, one has $\Phi_{(F;\mu)}(c)=0$ if and only if $\Psi_{(F;v_i)}(\mu)=0$, and we are done. □

Let $\mu \in \mathbb{R}$ be a real number and fix a point $(a,h) \in \mathbb{R} \times \mathbb{N}$.

Definition 2. — *We say that (a,h) belongs to the* red part with respect to μ *of the Newton polygon of $F(y)$ if $h \geq 1$ and either (a,h) is the vertex of $\mathscr{N}(F)$ with minimum ordinate or it belongs to a side of $\mathscr{N}(F)$ with inclination greater than μ.*

Notice that if the red part with respect to μ of $\mathscr{N}(F)$ is empty, then there are no generalized power series solution of $(F=0)$ of order greater than μ: the vertex (a,h) with minimum ordinate has $h=0$ and all the sides of $\mathscr{N}(F)$ have inclination less than or equal to μ, hence for $\gamma > \mu$, the polynomial $\Phi_{(F;\mu)}(\mathfrak{c})$ is a non-zero constant and by Corollary 2 the set $\mathrm{NIC}_1^*(F)$ is empty. The reciprocal is not true as Example 1 (page 65) shows.

Lemma 2. — *Let $(\mu_1, c_1, \ldots, \mu_k, c_k) \in \mathrm{NIC}_k^*(F)$, $\varphi = \sum_{j=1}^{k} c_j\, x^{\mu_j}$ and $F_{k+1}(y) = F(\varphi + y)$. The red part of $\mathscr{N}(F_{k+1}(y))$ with respect to μ_k nonempty.*

Proof. — Let $(\mu, c) \in \mathrm{NIC}_1^*(F)$ and consider $G = F(c\,x^{\mu} + y)$. The red part of the Newton polygon of G with respect to μ is not empty. To see this, let $v_0, \ldots, v_t$ be the vertices of $\mathscr{N}(F)$ ordered by decreasing ordinate and let v_k be the vertex with highest ordinate in $L(F;\mu) \cap \mathscr{N}(F)$. The ordinate of this v_k is greater than zero because otherwise $\Phi_{(F;\mu)}(\mathfrak{c})$ would be a nonzero constant, in contradiction with the fact that $\Phi_{(F;\mu)}(c) = 0$.

Returning to the main argument, given a monomial $M = x^{\alpha} y_0^{\rho_0} \cdots y_n^{\rho_n}$, one may write

$$M(c\,x^{\mu} + y) = x^{\alpha} \prod_{i=0}^{n} (c\,\mu^{i}\, x^{\mu} + y_i)^{\rho_i} = M + R, \tag{3}$$

where the points corresponding to the monomials of R have ordinate less than $|\rho|$ and belong to the line with inclination μ passing through $(\alpha, |\rho|)$. If w is the intersection of $L(F;\mu)$ with the axis of abscissas, then the cloud of points $\mathscr{P}(G)$ of G is contained in the positive convex hull of $\{v_0, \ldots, v_k, w\}$. The coefficient of G corresponding to w is precisely $\Phi_{(F;\mu)}(c) = 0$, hence $w \notin \mathscr{P}(G)$. Moreover, $\{v_0, \ldots, v_k\} \subseteq \mathscr{P}(G)$, because of (3). Therefore $v_0, \ldots, v_k$ are vertices of $\mathscr{N}(G)$. Hence either v_k is the vertex of $\mathscr{N}(G)$ with minimum ordinate or there exists a side of $\mathscr{N}(G)$ with inclination greater than μ and we are done. □

***Example 1* (See Figure 1)**. — *Let $F = x^{-1}\, y_0^6\, y_1 + y_0^2\, y_1 + x\, y_0^2 - 3\, x\, y_0\, y_1 - x^2\, y_0 + 2\, x^2\, y_1 + x^5$. The point $(1,1) \in \mathrm{NIC}^*(F)$. Let $G = F(x+y)$. The red part of $\mathscr{N}(G)$ with respect to $\mu = 1$ is vertex v_2' and point p. In this example, there are no solutions of $(G=0)$ of order greater than 1.*