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RESTRICTED VOLUMES OF EFFECTIVE DIVISORS

by Lorenzo Di Biagio & Gianluca Pacienza

Abstract. — We study the restricted volume of effective divisors, its properties and the relationship with the related notion of reduced volume, defined via multiplier ideals, and with the asymptotic intersection number. We build upon the fundamental work of Lazarsfeld and Mustăţa relating the restricted volume of big divisors to the volume of the associated Okounkov body. We extend their constructions and results to the case of effective divisors, recovering some results of Kaveh and Khovanskii, proving a Fujita-type approximation in this larger setting and studying the restricted volume function. In order to relate the reduced volume and the asymptotic intersection number we investigate a boundedness property of asymptotic multiplier ideals and prove it holds, for instance, for finitely generated divisors. In this way we obtain also a complete picture for the canonical divisor of an arbitrary smooth projective variety and for nef divisors on varieties of dimension at most 3.

Résumé (*Volumes restreints de diviseurs effectifs*). — Nous étudions le volume restreint de diviseurs effectifs, ses propriétés et la relation avec le volume réduit, défini en termes d'idéaux multiplicateurs, ainsi qu'avec le nombre d'intersection asymptotique. Nous nous basons sur le travail fondamental de Lazarsfeld et Mustăţa qui met en relation le volume restreint d'un diviseur gros avec le volume du corps d'Okounkov associé. Nous étendons leurs constructions et résultats au cas des diviseurs effectifs. Nous retrouvons en particulier certains résultats de Kaveh et Khovanskii, démontrons une approximation de Fujita dans ce cadre plus large et étudions la fonction volume

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restreint. Afin de relier le volume réduit et le nombre d'intersection asymptotique nous étudions une propriété d'encadrement des idéaux multiplicateurs asymptotiques et montrons qu'elle est valable, par exemple, dans le cas des diviseurs de type fini. De cette manière nous obtenons une description complète pour le diviseur canonique d'une variété lisse et projective quelconque et pour les diviseurs nef sur les variétés de dimension au plus 3.

1. Introduction

1.1. Motivations. — Let X be a smooth complex projective variety of dimension n and let D be a Cartier divisor. As it is well known, ample divisors display beautiful geometric properties, as well as cohomological and numerical ones. For long time it was thought that little, instead, could be said about general effective divisors. Anyway, at the beginning of this new century it became clear (see, e.g., [21], [17], [7]) that the classes of big divisors share some of these properties if one is willing to work asymptotically: if D is big, usually D^n does not carry immediate geometric information, while it is the *volume* of D, $\mathrm{vol}(D) := \lim_{m\to+\infty} \frac{h^0(X,\mathcal{O}_X(mD))}{m^n/n!}$, that is actually the right generalization of the top intersection number of ample divisors.

Let now V be a d-dimensional irreducible subvariety of X. If A is ample then the intersection number $A^d \cdot V$ plays an important role in many geometric questions. Along the previous lines, Ein *et al.* started in [9] a thorough study of asymptotic analogues of this degree for big divisors on V.
In case of an arbitrary divisor D, the *restricted volume* of D along V is

$$\mathrm{vol}_{X|V}(D) := \limsup_{m\to+\infty} \frac{\dim H^0(X \mid V, \mathcal{O}_X(mD))}{m^d/d!},$$

where $H^0(X \mid V, \mathcal{O}_X(mD)) := \mathrm{Im}\left(H^0(X, \mathcal{O}_X(mD)) \overset{\mathrm{restr}_V}{\to} H^0(V, \mathcal{O}_V(mD))\right)$. In the big case, restricted volumes have played an important role in the proof of boundedness of pluricanonical maps of varieties of general type (see [11], [26]), in the Fujita type results (see, e.g., [9, Theorem 2.20] and [26, Proposition 5.3]), as well as in the study of the cone of pseudoeffective divisors [2].

For an ample divisor A, since by Serre's vanishing theorem the restriction maps are eventually surjective, we have

$$\mathrm{vol}_{X|V}(A) = \mathrm{vol}_V(A_{|V}) = A^d \cdot V. \tag{1}$$

For an arbitrary divisor D on X, different generalizations of the intersection number have been studied in the recent literature (cf. [9] and [26]): apart from

$\mathrm{vol}_{X|V}(D)$, the *reduced volume* $\mu(V,D)$ of D along V, and the *asymptotic intersection number* $\|D^d \cdot V\|$ of D and V:

$$\mu(V,D) := \limsup_{m\to+\infty} \frac{\dim\left(H^0(\mathcal{O}_V(mD)\otimes \mathcal{J}(X,\|mD\|)_V)\right)}{m^d/d!},$$

$$\|D^d\cdot V\| = \limsup_{m\to+\infty} \frac{\sharp\,(V\cap D_{m,1}\cap\cdots\cap D_{m,d}\setminus \mathrm{Bs}(|mD|))}{m^d},$$

where $\mathcal{J}(X,\|mD\|)_V$ is the ideal in $\mathcal{O}_V$ generated by the asymptotic multiplier ideal $\mathcal{J}(X,\|mD\|)$ (see [18, Definition 11.1.2]) and $D_{m,1},\dots,D_{m,d}$ are general elements in $|mD|$. (See Section 3.2 and Section 5.1).

If D is big and V does not lie in a closed subset of X (which is precisely the augmented base locus of D, cf. [8] for the definition) then we have that $\mathrm{vol}_{X|V}(D) = \mu(V,D) = \|D^d\cdot V\|$ (see [9, Theorem 2.13] and [26, Theorem 3.1]). In particular the equality $\mathrm{vol}_{X|V}(D) = \|D^d\cdot V\|$ can be read as a generalization of Fujita's approximation theorem, and it leads to several interesting consequences (see [9, Corollary 2.14, Corollary 2.15, Corollary 2.16]). When the divisor D is effective, but not big, the lack of positivity seemed to be both a technical obstacle and a possible source of pathologies.

The second named author and Takayama have initiated, in [24], the study of the notions of restricted, reduced volume and asymptotic intersection number for line bundles L such that $0<\kappa(X,L)<\dim(X)$. Their main achievement is the proof that if V contains a (very) general point then $\mu(V,L)>0 \Leftrightarrow \mathrm{vol}_{X|V}(L)>0$. These two positivities are in turn equivalent to requiring that the Iitaka fibration f associated to L and restricted to V gives a generically finite map

$$f|_V : V \dashrightarrow f(V) \tag{2}$$

(see [24, Theorem 1.1, Corollary 1.2]). For such subvarieties the rate of growth of the dimensions of the restricted linear series is maximal and we are therefore in a situation similar to the big case.

The purpose of this paper is to show that indeed restricted volumes of effective divisors along subvarieties verifying condition (2) (which turns out to be an optimal condition) share the same nice properties enjoyed by restricted volumes of big divisors. Moreover, under the same condition, the relationships among the different variants are well understood, modulo a property, which holds for instance for finitely generated divisors on arbitrary varieties (hence for the canonical divisor) or in low dimension.