Revue d'histoire des mathématiques
11 (2005), p. 7–21

THE AREA AND THE SIDE I ADDED: SOME OLD BABYLONIAN GEOMETRY

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Abstract. — There was a standard procedure in Mesopotamia for solving quadratic problems involving lengths and areas of squares. In this paper, we show, by means of an example from Susa, how area constants were used to reduce problems involving other geometrical figures to the standard form.

Résumé (La surface et le côté que j'ai ajouté : un problème de géométrie babylonienne)

Il y avait en Mésopotamie un procédé standard pour résoudre des problèmes quadratiques impliquant des longueurs et des surfaces de carrés. Nous montrons, sur un exemple de Suse, que des constantes géométriques ont été employées pour ramener des problèmes concernant d'autres figures au format standard.

INTRODUCTION

One of the central topics of Old Babylonian mathematics is the solution of 'quadratic' or area problems. Høyrup has developed a convincing geometric interpretation for the procedures by which many of these problems were solved. However, it has not previously been recognized how Old

Texte reçu le 17 novembre 2003, révisé le 23 août 2004.

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2000 Mathematics Subject Classification : 01A17.
Key words and phrases : Old Babylonian, Susa, area constants, apsamikkum.
Mots clefs. — Mésopotamie, Suse, constantes géométriques, apsamikkum.

Babylonian scribes applied these techniques when the underlying figure was not a square. Here, we show by means of an exemplar from Susa that the solution to this problem lays in an elegant usage of another ubiquitous element of Old Babylonian mathematics, the technical constant, or coefficient.

Among the corpus of Old Babylonian geometric problems is a group of exercises that begin with variations on the phrase, "The area and side of my square I added..." The resulting total is given, and the problem for the student is to determine the length of the side of the square.[1] On first translation, these problems were seen as exercises in pure algebra, dressed in physical guise. If the unknown length of the side of the square is denoted ℓ, then the sum of the area and side is $\ell^2 + \ell$, and the goal of the problem is to solve for the side ℓ. On the other hand, sides and squares are geometrical objects, and if treated geometrically, then addition of lengths and areas makes no sense: the equations are not homogeneous. If Mesopotamian scribes of the second millennium are not to be considered to have had a concept of abstract algebra, a conceptual development that lay far in their future, then the geometrical viewpoint appears to present an insurmountable hurdle.

The resolution of this apparent dilemma has been developed by Høyrup over the last twenty years or so, culminating in the exposition in [Høyrup 2002]. By a close reading of the specific mathematical terminology involved, Høyrup showed that the stumbling block for a modern understanding of ancient geometry lay in our inheritance of the Greek categories of inhomogeneous lines and areas, and specifically in the Euclidean notion that a line has no breadth. Høyrup has convincingly demonstrated that in the pre-Euclidean world of the Old Babylonian scribes, lines are best understood as having unit breadth. That is, in algebraic contexts, we should view the side ℓ as in fact an area $1 \cdot \ell$, where

[1] The phrase itself occurs in the first problem of BM 13910 (discussed below) as 'a.šà^{lam} ù *mi-it-har-ti ak-mur-ma*', which Høyrup renders as "the surface and my confrontation I have accumulated" [Høyrup 2002, p. 50]. The rest of the problems on BM 13901 are related – there are assorted additions, subtractions and combinations. Similar problems are to be found in AO 6770, AO 8862, BM 85200 + VAT 6599, TMS 8, TMS 9, TMS 16, and YBC 6504. Additionally, BM 80209 contains statements of similar problems involving circles, but with no procedure given, and TMS 20, the main text discussed here, contains both statements and procedures for solving the problem in the case of the *apsamikkum* (see below).

the factor 1 is usually hidden from view. In geometric terms, the length added to the area becomes a rectangle of length 1 adjoined to a square.

More than the statement of the problem, it is a detailed analysis of the individual steps of the solution procedure that provides the most forceful evidence of the persuasiveness of Høyrup's case. His approach is based upon a very deep understanding of the precise usage of the technical vocabulary of Old Babylonian mathematics and a fine judgment for the careful way it is employed. It is clear from his work that Old Babylonian categories of thought relating to mathematical operations were not the same as ours and that in many cases, they made finer distinctions than our more abstract approach allows.[2] Høyrup has developed an English vocabulary to reflect these fine distinctions, but we will simplify some of the terminology here when the shades of meaning do not affect our argument. We also stress that Høyrup's analysis goes far beyond the problems considered here.

THE SIDE AND THE SQUARE

The classic example of this approach, explained many times by Høyrup himself and numerous other commentators is the first problem on the Old Babylonian tablet BM 13901.[3] The problem reads as follows:

The area and side of my square I added: 0;45.[4]

[2] For example, Høyrup distinguishes four groups of terms for what we consider multiplication. Briefly, these may be described as: the a.rá or 'steps of' from the multiplication tables; terms derived from *našūm*, 'to raise or lift' for multiplication by constants, etc.; *espum*, a doubling or more general repetition, and terms based on *šutakūlum*, which generate physical areas from bounding lengths and widths. For more details see [Høyrup 2002, pp. 18–40] in general and [Muroi 2002] on terms for multiplication.

[3] The text was first published by Thureau-Dangin [1936] and re-edited by Neugebauer [1935/37]. It is of unknown provenance and currently housed in the British Museum. Høyrup includes in [Høyrup 2002] many of the problems (Problems 1, 2, 3, 8, 9, 10, 12, 14, 18, 23, 24) and has published a complete edition in [Høyrup 2001]; his transliteration mostly follows that of Neugebauer, with a few minor differences in restoration of broken passages. I have followed Høyrup's transliteration, but the translation below is mine.

[4] Abstract numbers in Old Babylonian mathematics were written in a sexagesimal, or base 60, place-value system. The value of a sign depended on the sexagesimal 'column' in which it occurred. However, these numbers contained no explicit reference to absolute size in terms

You, put down 1, the projection.
Break 1 in half.
Multiply 0 ; 30 and 0 ; 30.
Join 0 ; 15 to 0 ; 45: 1.
1 is the square root (of 1).
Subtract the 0 ; 30 which you multiplied in the 1: 0 ; 30.
The side (is) 0 ; 30.

Before illustrating Høyrup's geometric interpretation of this problem, it is worth giving a formal analysis of the steps of the algorithm for comparison with later problems. This approach is adapted from Ritter, who introduced a similar technique for comparing medical, divinatory and jurisprudential texts with mathematical ones in Egypt and Mesopotamia in [Ritter 1995a;b; 1998]. Ritter's main concern was to show that divination, medicine and mathematics formed a common intellectual domain for Old Babylonian scribes. Of course, the technical vocabulary of these three areas is quite discipline-specific, so that whereas Høyrup has focused on the terminology of mathematics, Ritter was drawn to analyzing the underlying grammatical and organizational structure of the texts. In particular, he showed the consistent ways in which grammatical signifiers marked off sections of the texts. As with Høyrup, Ritter's analysis is much wider and deeper than the specific cases considered here.

In order to analyze the underlying structures of Old Babylonian mathematical procedures, Ritter introduced a 'schematic form' of representation to show the linkages between individual arithmetic steps. The example he chose in [Ritter 1995a] to illustrate this technique was BM 13901, giving a full analysis of the first three problems and an abbreviated description of the remainder. As part of his analysis, Ritter showed that there are some core, basic algorithms in Mesopotamian (and Egyptian) mathematics, as well as many variations on that core. Ritter [1998] returned

of the base unit. In order to align columns correctly for addition, size had to be inferred from context. There are several conventions for transliterating cuneiform numbers. Here, we follow the widely-used Neugebauer convention where each sexagesimal place is separated by a comma, and the 'sexagesimal point' separating multiples from fractions is denoted by a semi-colon. That is, 1,30 represents $1 \times 60 + 30 = 90$, but 1 ; 30 represents $1 + \frac{30}{60} = 1\frac{1}{2}$. For convenience of the reader, we have silently introduced an absolute size for the problems we discuss, but we stress that this notation is not present in the cuneiform original.

	Computation	*Symbolic Instruction*
Step 0	Put down 1	w
Step 1	Break 1 in half: 0 30	$R_1 := \frac{1}{2}w$
Step 2	Multiply 0 30 and 0 30: 0 15	$R_2 := R_1^2$
Step 3	Join 0 15 to 0 45: 1	$R_3 := R_2 + D$
Step 4	Square root of 1: 1	$R_4 := \sqrt{R_3}$
Step 5	Subtract 0 30 from 1: 0 30	$R_5 := R_4 - R_1$

TABLE 1. BM 13901, Problem 1

to the same example when comparing the structure of the algorithm to that of the problem in Str. 368. The notation he used in the latter paper was somewhat different to the earlier version; it is this later technique we have adapted here. As some of the points we wish to emphasize are a bit different to Ritter's focus, we have changed the notation somewhat. Ritter's approach has also been extensively used by Imhausen [2002; 2003] in studying Egyptian mathematics.

In this formalism, three types of information are differentiated. First, there are the data that are explicitly given in the statement of the problem; these data are denoted $D_1, \ldots, D_m$. In the case of BM 13901, Problem 1, the only datum given is the total area, which we denote by $D = 0;45$. Secondly, there is the implicit data, in this case the projection, which always has unit length, this we denote by $w = 1$. Finally, each arithmetical step of the algorithm produces a result, and we denote the result of step n by R_n. Using this approach, we may present the problem above in the form of Table 1.

One advantage of this particular approach to Old Babylonian mathematics is that it helps to foreground the fact that, as far as possible, each step of the algorithm uses the result of the immediately preceding step as one of its inputs, so that Step 4 uses R_3 (the result of Step 3), Step 3 uses the result of Step 2, and Step 2 uses the result of Step 1. The final Step 5 uses the result of the preceding step as well as the result of an earlier step, here Step 1. This important characteristic is lost in an algebraic description of the problem as an equation such as $x = \sqrt{(\frac{1}{2}w)^2 - \frac{1}{2}w}$.