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INDEX THEOREM FOR ELLIPTIC PAIRS

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Introduction

In this series of papers, we investigate the relative index theorem in the framework of algebraic analysis.

On a complex manifold X, let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and F an $\mathbb{R}$-constructible sheaf (for the underlying real analytic structure of X). The complex

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{O}_X))$$

is the complex of solutions of the system of PDE represented by $\mathcal{M}$ in the sheaf of generalized holomorphic functions associated to F. For example, if X is the complexification of a real analytic manifold M, and $F = \mathrm{or}_M$, we get the complex of Sato's hyperfunction solutions, or else if F is $\mathbb{C}$-constructible, we find a complex of ramified holomorphic solutions.

A natural problem is to find conditions under which such a complex has finite dimensional global cohomology and then to compute the corresponding Euler-Poincaré characteristic.

In our first paper, we prove the finiteness theorem when $(\mathcal{M}, F)$ has compact support and is "elliptic", i.e.:

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X$$

where $\mathrm{char}(\mathcal{M})$ is the characteristic variety of $\mathcal{M}$, $SS(F)$ is the micro-support of F and $T^*_X X$ is the zero section of the cotangent bundle.

In fact, we give a relative version of this finiteness result together with the associated duality theorem and Künneth formula. Our methods rely upon results of functional analysis over a sheaf of Fréchet algebras which are developped in the last paper of this volume.

With finiteness, duality and Künneth formula at hand, we have all the basic tools needed to get an index formula along the line of the Lefschetz fixed point theorem. Such an approach is developped in our second paper. We attach a "microlocal Euler class"

$$\mu\mathrm{eu}(\mathcal{M}, F) \in H^{2\dim X}_{\mathrm{char}\mathcal{M}+SSF}(T^*X; \mathbb{C})$$

to any elliptic pair $(\mathcal{M}, F)$ and prove that, under natural assumptions, this class is compatible with direct images, inverse images and external products. In particular, it is the microlocal product of a class $\mu\mathrm{eu}(\mathcal{M})$ attached to $\mathcal{M}$ and a class $\mu\mathrm{eu}(F)$ attached to F, this last one being nothing but the Kashiwara's Lagrangian cycle of F. We also give the index formula:

$$\chi(\mathrm{R}\Gamma(X; R\mathcal{H}om(\mathcal{M} \otimes F; \mathcal{O}_X))) = \int \mu\mathrm{eu}(\mathcal{M}, F)_{|T^*_X X} = \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(F).$$

Note that $(\mathcal{M}, \mathbb{C}_X)$ is always elliptic. Hence our results contain many results of $\mathcal{D}$-module theory. Moreover, choosing $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{G}$ for a coherent $\mathcal{O}_X$-module $\mathcal{G}$ allows us to recover classical results of analytic geometry.

When $F = \mathbb{C}_M$, our results for the pair $(\mathcal{M}, F)$ give an index theorem for elliptic systems and we discuss its relations with the Atiyah-Singer theorem.

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Elliptic pairs I. Relative finiteness and duality

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Elliptic Pairs I. Relative Finiteness and Duality

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Contents

1 Introduction

Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds, $\mathcal{M}$ a coherent module over the ring $\mathcal{D}_X$ of differential operators on X, F an $\mathbb{R}$-constructible object on X. In this first paper, we give a criterion insuring that the derived direct images of the $\mathcal{D}_X$-module $F \otimes \mathcal{M}$ are coherent $\mathcal{D}_Y$-modules, and we prove related duality and Künneth formulas. Part of these results were announced in [20, 21].

In [22], making full use of these results, we shall associate to ($\mathcal{M}$,F) a characteristic class and show its compatibility with direct image, thus obtaining an index theorem generalizing (in some sense) the Atiyah-Singer index theorem as well as its relative version [1, 3].

Let us describe our results with more details, beginning with the non-relative case for the sake of simplicity.

An elliptic pair on a complex analytic manifold X is the data of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and an $\mathbb{R}$-constructible sheaf F on X (more precisely, objects of the derived categories), these data satisfying the transversality condition

$$\operatorname{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X. \tag{1.1}$$

Here $\operatorname{char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$, $SS(F)$ the micro-support of F (see [12]) and $T^*_X X$ the zero section of the cotangent bundle T^*X.

This notion unifies many classical situations. For example, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then the pair ($\mathcal{M}$,$\mathbb{C}_X$) is elliptic. If U is an open subset of X with smooth boundary ∂U, the pair ($\mathcal{M}$,$\mathbb{C}_U$) is elliptic if and only if ∂U is non characteristic for $\mathcal{M}$. If X is the complexification of a real analytic manifold M, then ($\mathcal{M}$,$\mathbb{C}_M$) is an elliptic pair if and only if $\mathcal{M}$ is elliptic on M in the classical sense. If F is $\mathbb{R}$-constructible on X, then ($\mathcal{O}_X$,F) is an elliptic pair. If $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module, we can associate to it the coherent $\mathcal{D}_X$-module $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, and the results obtained for the elliptic pair ($\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$,$\mathbb{C}_X$) will give similar results for $\mathcal{G}$. See §8 for a more detailed discussion.

If $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds, we generalize the preceding definition and introduce the notion of an f-elliptic pair, replacing in (1.1) $\operatorname{char}(\mathcal{M})$ by $\operatorname{char}_f(\mathcal{M})$, the f-characteristic variety of $\mathcal{M}$ (this set was already defined in [19] when f is smooth).

The main results of this paper assert that if the pair ($\mathcal{M}$,F) is f-elliptic, f is proper on $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(F)$ and $\mathcal{M}$ is endowed with a good filtration, then:

1) the direct image (in the sense of $\mathcal{D}$-modules) $\underline{f}_!(\mathcal{M} \otimes F)$ has $\mathcal{D}_Y$-coherent cohomology,

2) the duality morphism

$$\underline{f}_!(D'F \otimes \underline{D}_X\mathcal{M}) \longrightarrow \underline{D}_Y \underline{f}_!(\mathcal{M} \otimes F)$$

is an isomorphism (here, $\underline{D}$ denotes the dualizing functor for $\mathcal{D}$-modules and D' is the simple dual for sheaves),