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THE SIGNATURE OPERATOR ON MANIFOLDS WITH A CONICAL SINGULAR STRATUM

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60[th] birthday

Abstract. — We consider a Riemannian manifold, M, which can be compactified by adjoining a smooth compact oriented Riemannian manifold such that a neighbourhood of the singular stratum B, of codimension at least two, is given by a family of metric cones. Under the assumption that the middle cohomology of the cross-section vanishes, we show that there is a natural self-adjoint extension for the Dirac operator on forms with discrete spectrum, and we determine the condition of essential self-adjointness. We describe the boundary conditions analytically and construct a good parametrix which leads to the asymptotic expansion of a suitable resolvent trace as in our previous work. We also give a new proof of the local formula for the L^2-signature.

Résumé **(Opérateur de signature sur les variétés avec une strate singulière conique)**
Nous considèrons une variété riemannienne M, qui peut être compactifiée en lui adjoignant une variété riemannienne C^∞ compacte orientée, telle qu'un voisinage de la strate singulière B, de codimension au moins deux, est donné par une famille de cônes métriques. Sous une hypothèse d'annulation de la cohomologie de la section du cône en dimension moitié, nous montrons qu'il existe une extension auto-adjointe naturelle de l'opérateur de Dirac agissant sur les formes qui est de spectre discret, et nous déterminons la condition sous laquelle l'opérateur de Dirac est essentiellement auto-adjoint. Nous décrivons les conditions de bord, et nous construisons une parametrix qui donne le développement asymptotique de la trace de la résolvante, comme dans un travail antérieur. Nous donnons aussi une preuve nouvelle de la formule locale pour la signature L^2.

Introduction

In this article, we analyze the signature operator on an oriented Riemannian manifold (M, g), of dimension $m = 4k$, with one compact singular stratum B of dimension

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h (the "horizontal dimension"), such that $m-h \geq 2$. A neighbourhood of the singular set is given by

$$U := U_{\varepsilon_0} := (0,\varepsilon_0) \times N, \ \varepsilon_0 \in (0,1/2), \tag{0.1}$$

with an oriented compact Riemannian manifold N of dimension $4k-1$ and metric g^{TN}, and M decomposes as

$$M =: U_{\varepsilon_0} \cup M_{\varepsilon_0} \tag{0.2}$$

into points of distance at most and at least ε_0 of the singular set, respectively. For $\varepsilon \in (0,\varepsilon_0]$, we use analogous notation and write $U_\varepsilon, M_\varepsilon$, with

$$M = U_\varepsilon \cup M_\varepsilon.$$

We assume that the orientation on M and N induce the boundary orientation on U, such that $\{-\frac{\partial}{\partial t}, e_1, \dots, e_{m-1}\}$ is oriented on U if $t \in (0,\varepsilon_0)$ and $\{e_1,\dots,e_{m-1}\}$ is oriented on N. We assume in addition that the singularity is of the following special type. There is a fibration of oriented compact Riemannian manifolds,

$$\pi : Y \hookrightarrow N \to B, \tag{0.3}$$

with fibers $Y_b = \pi^{-1}(b)$, $b \in B$, of dimension $v := 4k-1-h \geq 1$ (the "vertical dimension"); in particular, B carries a metric g^{TB} such that π becomes a Riemannian submersion. Then the tangent bundle TN of N splits under g^{TN} into the *vertical* and the *horizontal* tangent bundle, consisting of the tangent vectors to the fibers and their orthogonal complement,

$$TN_p \quad =: \quad T_H N_p \oplus T_V N_p, \tag{0.4}$$

with induced metrics $g^{T_H N}$ and $g^{T_V N}$; the corresponding orthogonal projections in TN will be denoted by P_H and P_V, respectively. Next we assume that the metric $g^{TU} := g^{TM}|U$ takes the form

$$g^{TU} := dt^2 \oplus g^{T_H N} \oplus t^2 g^{T_V N}, \tag{0.5}$$

which we will call a metric of *conic type.* Thus, $M \cup B$ is a Riemannian pseudomanifold with one singular stratum of conic type.

The boundary of U is the Riemannian manifold

$$N_{\varepsilon_0} := (N, g^{TN}_{\varepsilon_0} := g^{T_H N} \oplus \varepsilon_0^2 g^{T_V N}). \tag{0.6}$$

The splitting of TN induces a splitting of the cotangent bundle,

$$T^*N =: T^*_H N \oplus T^*_V N,$$

into cotangent vectors annihilating $T_V N$ and $T_H N$, respectively. This splitting induces a bigrading of the exterior algebra $\Lambda T^* N$ which will be important for our analysis; we write

$$\begin{aligned} \Lambda T^*N &= \Lambda T^*_H N \otimes \Lambda T^*_V N \\ &= \oplus_{j=p+q} \Lambda^p T^*_H N \otimes \Lambda^q T^*_V N =: \oplus_{p,q} \Lambda^{p,q} T^* N. \end{aligned} \tag{0.7}$$

The smooth sections of ΛT^*N and $\Lambda T^*_{H/V}N$ will be denoted $\lambda(N)$ and $\lambda_{H/V}(N)$, respectively, with degree or bidegree noted with superscripts.

Our main object will be the canonical Dirac operator associated with ΛT^*M,

$$D^\Lambda := D^\Lambda_M := d_M + d^\dagger_M, \tag{0.8}$$

with $d_M =: d$ the exterior derivative on M and $d^\dagger$ its formal adjoint with respect to the metric g^{TM}.

D defined on forms with compact support, denoted by $\lambda_c(M)$, is symmetric in $L^2(M, \Lambda T^*M) =: \lambda_{(2)}(M)$ but may not be essentially self-adjoint; we refer to the closure of this operator as $D^\Lambda_{\min} =: D_{\min}$, and $d_{\min}, d^\dagger_{\min}$ are defined analogously.

A specific self-adjoint extension of this operator can be defined via the Hilbert complex given by the operator $d_{\max}$ which arises from $d^\dagger$ as

$$d_{\max} := (d^\dagger_{\min})^*, \tag{0.9}$$

cf. [**11**, §3]; with a slight abuse of notation we denote this extension again by $D = D^\Lambda = D^\Lambda_M$, with domain $\mathcal{D} = \operatorname{dom} D$. In general, there will be many more self-adjoint extensions but D is of interest since its kernel gives the L^2-cohomology of M. If D is a Fredholm operator we have to break its symmetry to obtain a nontrivial index, e. g. by an anticommuting *supersymmetry* i. e. a self-adjoint involution of ΛT^*M. We will use multiplication by the complex volume element, τ_M, which splits

$$\Lambda T^*M =: \Lambda^+T^*M \oplus \Lambda^-T^*M$$

into ± 1-eigenbundles and analogously

$$\lambda(M) =: \lambda^+(M) \oplus \lambda^-(M),$$

with associated splitting $\sigma = \sigma^+ + \sigma^-$ on the level of forms. If τ_M maps $\mathcal{D}$ to itself than we can define the *Signature Operator* of M, with domain $\mathcal{D}^{\mathrm{sign}} = \mathcal{D}^+ = \frac{1}{2}(I + \tau_M)\mathcal{D}$, by

$$D^{\mathrm{sign}}_M := D^{\mathrm{sign}} := D^\Lambda_M | \mathcal{D}^+ : \mathcal{D}^+ \to \mathcal{D}^-. \tag{0.10}$$

We say that the *case of uniqueness* or the L^2-Stokes Theorem holds on M if

$$d_{\max} = d_{\min}. \tag{0.11}$$

In this case we have $\tau(\mathcal{D}) \subset \mathcal{D}$, and if D is also Fredholm, then so is D^{sign} and its index equals the L^2-signature of M,

$$\operatorname{ind} D^{\mathrm{sign}} = \operatorname{sign}_{(2)} M. \tag{0.12}$$

The above metric data define the crucial object in the analysis of the signature operator: the splitting of T^*N (induced by (0.4)) defines the "vertical de Rham operator" d_V (see (2.5)) and the metric $g_1^{T_VN}$ defines the adjoint $d_V^\dagger$, such that we can form the operator (see (2.31))

$$A_V := (d_V + d_V^\dagger)\alpha + \nu. \tag{0.13}$$

Here α is another supersymmetry on ΛT^*N and ν is an endomorphism (which are defined in (2.19) and (2.13)), and A_V is a first order symmetric differential operator on $C_c^1(N, \Lambda T^*N)$ which is fiberwise elliptic. Now M is called a *Witt space* if

$$H^{v/2}(Y) = 0. \tag{0.14}$$

We will see below (cf. Theorem 3.1) that (0.14) is essentially equivalent to the analytic condition

$$A_V \text{ is invertible,} \tag{0.15}$$

in the sense that the invertibility of A_V implies the Witt condition, whereas the Witt condition does not exclude the existence of zero eigenvalues but only of such which may be called inessential; indeed, they disappear under suitable rescalings of the fiber metric. *We will assume that M is a Witt space.*

Our results can then be summarized in the following theorems. We describe the Signature Operator on M by explicitly constructing its Green kernel which relates it to the symmetric operator $\tilde{D}$ defined as the restriction of $D_{\max}$ to the domain

$$(0.16) \quad \{\sigma \in \operatorname{dom} D_{\max} : ||\sigma^+||_{\lambda_{(2)}(N_t)} = O(t^{1/2-\varepsilon}) \text{ for every } \varepsilon > 0, \\ ||\sigma^-||_{\lambda_{(2)}(N_t)} = O(t^{-1/2+\eta}) \text{ for some } \eta > 0, t \to 0\};$$

note that $\tilde{D}$ anticommutes with τ_M by construction.

Theorem 0.1. — *Let the Riemannian manifold (M, g^{TM}), of dimension $m = 4k$, be the top stratum of a Riemannian pseudomanifold, X, which is a Witt space with only one singular stratum B of conic type.*
1. *The operator $\tilde{D}$ defined by* (0.16) *is self-adjoint and discrete and anticommutes with τ_M.*
2. *If $|A_V| \geq \frac{1}{2}$, then $D^{\Lambda}_{M,\min}$ is essentially self-adjoint.*
3. *The case of uniqueness holds for M.*
4. $D^{\text{sign}} = \tilde{D}^+$.

This theorem is well known in the case $h = 0$, cf. [**15**], [**12**], and part 2 and part 3 could also be deduced from Cheeger's work [**15**].

It is clear from part 4 of Theorem 0.1 that under the above conditions

$$\operatorname{ind} \tilde{D}^+ = \operatorname{ind} \tilde{D}^{\text{sign}} = \operatorname{sign}_{(2)} M, \tag{0.17}$$

so it is natural to ask for a local formula analogous to Hirzebruch's Signature Theorem in the smooth case. Bismut and Cheeger [**6**, Thm. 5.7] have indicated the adiabatic construction of the homology L-class on the compact singular space associated with M, together with the corresponding L^2-index formula. A crucial role is played by the η-invariant, $\eta(N, g^{TN})$, of the Riemannian manifold (N, g^{TN}), as introduced by Atiyah, Patodi, and Singer in [**1**, Thm. (4.14)], and its adiabatic limit,

$$\tilde{\eta}(N, g^{TN}) := \lim_{\epsilon \to 0} \eta(N, g_\varepsilon^{TN}).$$

The adiabatic limit was first introduced and computed by Witten [**25**], as a gravitational anomaly, in case of a one-dimensional base. Witten's formula was proved rigorously by Cheeger [**16**], and independently by Bismut and Freed [**9**], [**10**]. The computation of the adiabatic limit for arbitrary dimensions and invertible fiber operators was given by Bismut and Cheeger [**6, 7**], who introduced the form $\tilde{\eta} = \tilde{\eta}(\pi, g^{TM}) \in \lambda(B)$ generalizing the η-invariant; the case of the signature operator was treated by Dai [**18**, Thm.0.3] who further introduced the τ-invariant associated to the Leray spectral sequence of the fibration (0.3). There has been done considerable work recently on the computation of L^2-cohomology groups of spaces which can be compactified as pseudomanifolds of the type we consider here, cf. [**19**], [**20**], [**21**], and [**17**]. These calculations lead to topological formulas for $\operatorname{sign}_{(2)} M$, see [**17**, Cor.1.2] for Witt spaces and its extension in [**21**]. Combining these topological formulas with Dai's result quoted above gives the following local signature formula which was stated for even dimensional base spaces in [**8**, Thm.5.7]; in its formulation, we denote by $D_B^{\Lambda\otimes\mathscr{H}(Y)}$ the Dirac operator D_B^{Λ} twisted by the bundle of fiber harmonic forms.

Theorem 0.2. — *We have*

$$\operatorname{ind} D^{\text{sign}} = \lim_{\varepsilon\to 0} \int_M L(\nabla^{TM}) - \int_B L(TB, \nabla^{TB}) \wedge \tilde{\eta} - \frac{1}{2}\eta(D_B^{\Lambda\otimes\mathscr{H}(Y)}).$$

We give here an analytic proof of [**17**, Cor.1.2] in the general case which should be applicable to more general situations; in combination with the results of Atiyah, Patodi, and Singer and Dai's computation, it yields the theorem. The parametrix construction which we give in this paper should, in principle, also lead directly to the local index formula but, so far, we have been unable to overcome the technical difficulties involved.

We also have considered the resolvent trace expansion. We have a proof of the following result, but its presentation would lengthen the paper unduly; we hope to include it in a more general result at some future time.

Theorem 0.3. — 1. *For $\mu \in \mathbb{R} \setminus \{0\}$ and $p > m$, the resolvent $(D - i\mu)^{-1}$ is in the Schatten-von Neumann class of order p in $L^2(M, \Lambda T^*M)$.*
2. *For $z \in \mathbb{R}$ and $l > m/2$, we have the asymptotic expansion*

$$\operatorname{tr}[D^2 + z^2]^{-l} \sim_{z\to\infty} z^{m-2l} \sum_{j\geq 0} a_j z^{-j} + \sum_{j\geq 2l-h} b_j z^{-j} \log z.$$

The plan of the article is as follows. In Section 1, we deal with general Dirac operators and derive some decomposition theorems which are induced by a fibration of the form (0.3) and are needed later on. These results are known for spin Dirac operators, see [**5**, pp. 56, 59].

In Section 2, we represent the signature operator D^{sign} on U in the form

$$D_M^{\text{sign}} \simeq \frac{\partial}{\partial t} + A_H(t) + t^{-1} A_V,$$

acting on $C_c^1\big((0,\varepsilon_0), H^1(N,\Lambda T^*N)\big)$ (see (2.38)). Here $A_H(t)$ and A_V are first order differential operators which can be written as a Dirac operator plus a potential and $A_H(t)$ is linear in t, with derivative a bounded endomorphism, while A_V is given by (0.13). We also show (in Theorem 2.5) that the anticommutator $A_H A_V + A_V A_H$ is a first order vertical differential operator, a crucial fact for our analysis. The guiding principles here are the structure of Dirac systems, as developped in [**3**], and the decomposition results from Sec. 1.

In Section 3, we obtain explicitly the spectral decomposition of the operators $A_V(b) := A_V|Y_b$ (cf. Theorem 3.1). By ellipticity, the spectrum is discrete. It consists of the *harmonic* eigenvalues $\mu = j - v/2, 0 \le j \le v$, generated by the harmonic forms on Y_b, and two families $\mu^\pm$ generated by the nonzero eigenvalues of the Laplacian on Y_b, with $\mu^+ \subset (-\frac{1}{2}, \infty)$ and $\mu^- \subset (-\infty, \frac{1}{2})$. When the metric on Y_b is scaled down, these eigenvalues tend respectively to $+\infty$ and $-\infty$.

Section 4 introduces appropriate boundary conditions for $D^{\rm sign}$, based on the spectral analysis of Section 2. For the choice of boundary conditions and hence of a self-adjoint extension, only the small eigenvalues of A_V matter. We treat them by explicitly constructing the resolvent kernel by means of matrix Bessel functions, as introduced in [**13**], and then use this kernel in constructing a good pseudodifferential parametrix for $D^{\rm sign}$ with operator valued symbol, again following the strategy developed in [**13**]. At the end of this section, we give the proof of Theorem 0.1.

In Section 5 we prove Theorem 0.2 by reducing the problem to an APS-type problem on M_ε, for sufficciently small $\varepsilon > 0$. We also prove various related results: a Kato type perturbation result for the APS projection (Theorem 5.9), a vanishing result which is crucial for our approach (Theorem 5.2), and a new identity involving Dai's τ-invariant (Theorem 5.4).

This paper started as a joint project with Bob Seeley to whom it owes a lot. The construction of the Signature Operator was essentially finished several years ago using a less explicit parametrix construction. The publication of the results has been delayed by an attempt to deduce the local signature formula directly from the resolvent expansion in Theorem 0.3. However, this goal has proved elusive so far; we hope that, nevertheless, the results presented here will be of independent value.

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1. Dirac operators on fibrations

In this section, we consider a Riemannian manifold (M, g^{TM}) which we assume to be oriented. For $X, Y \in TM$ we write

$$g^{TM}(X,Y) =: \langle X, Y\rangle_{TM} =: \langle X, Y\rangle,$$

if no confusion may arise, and we use similar notation for vector bundles. Moreover, we consider a second oriented Riemannian manifold (B, g^{TB}) and a Riemannian fibration

$$\pi = \pi_B^M : M \to B \tag{1.1}$$

with generic fiber F; we write

$$F_b := \pi^{-1}(b),\ \ b \in B. \tag{1.2}$$

We denote the bundle of tangent vectors to the fibers by $T_V M$. Then the fibration induces an orthogonal splitting

$$TM =: T_H M \oplus T_V M,\ \ g := g^{TM} =: g^{T_H M} \oplus g^{T_V M} =: g_H \oplus g_V,$$

with orthogonal projections $P_{H/V} : TM \to T_{H/V}M$. Note that $T_V M$ and its annihilator $T_H^* M$ are defined independent of the metric.

The bundle (TM, g^{TM}) has a distinguished metric connection, the Levi-Civita connection ∇^{TM}; all bundles associated to the principal bundle of orthonormal frames in TM inherit a metric and a metric connection from (TM, g^{TM}). This holds in particular for the exterior algebra of the cotangent bundle, $\Lambda T^* M$, and for the bundle of Clifford algebras, $Cl(TM)$, and its complexification, $\mathbb{C}l(TM) = Cl(TM) \otimes_{\mathbb{R}} \mathbb{C}$.

We are interested in the class of *Dirac bundles* as defined in [**23**, p. 114], i.e. the smooth hermitian bundles (E, h^E) over M equipped with hermitian connections ∇^E such that the following conditions are satisfied: There is a smooth bundle map cl from the tangent bundle, TM, to the skew-hermitian endomorphisms, $\mathrm{End}_{\mathrm{as}}\, E$, of E such that

$$\mathrm{cl}(X) \circ \mathrm{cl}(X) = -g(X,X) I_E,\ \ X \in TM, \tag{1.3}$$

which implies that cl extends to an algebra homomorphism

$$\mathrm{cl} : \mathbb{C}l(TM) \to \mathrm{End}\, E, \tag{1.4}$$

turning E into a left Clifford module. Moreover, ∇^E is required to be compatible with the Levi-Civita connection in the sense that

$$\nabla^E_X \mathrm{cl}(Y)\sigma = \mathrm{cl}(\nabla^{TM}_X Y)\sigma + \mathrm{cl}(Y)\nabla^E_X \sigma, \tag{1.5}$$

for $X, Y \in TM, \sigma \in C^1(M, E)$. A prototypical Dirac bundle is, of course, $\mathbb{C}l(TM)$ itself with the metric structure induced from g^{TM}. This bundle is canonically isomorphic to the exterior algebra bundle $\Lambda T^* M$, with Clifford action

$$\mathrm{cl}(X)\omega = \mathrm{w}(X^\flat)\omega - \mathrm{i}(X)\omega,\ \ X \in TM,\ \omega \in \Lambda T^* M,$$

where "w" and "i" refer to wedge and interior multiplication, respectively, while $\flat : TM \to T^*M$ denotes the "musical" isomorphism induced by g^{TM} with inverse $\sharp$. Note that these definitions extend naturally to Hilbert bundles over M.

The notion of Dirac bundle was introduced to define the *Dirac operator* naturally associated with it, i.e. the operator

$$D := D_M^E := \sum_{i=1}^m \mathrm{cl}(e_i)\nabla_{e_i}^E, \tag{1.6}$$

which we will regard as an unbounded operator in $L^2(M, E)$ with domain $C_c^1(M, E)$ if not stated otherwise. Then D is symmetric in $L^2(M, E)$ and essentially self-adjoint e.g. if M is complete.

To obtain a nontrivial index, the symmetry of D must be broken. This is achieved by a *supersymmetry* or grading, α, on E, i.e. by a self-adjoint involution $\alpha \in \mathrm{End}\, E$ which is parallel with respect to ∇^E and anticommutes with Clifford multiplication, and hence with D. Then the bundle E splits as

$$E = E^+ \oplus E^-, \ E^\pm = \frac{1}{2}(I \pm \alpha)E.$$

$\mathbb{C}l(TM)$ has a natural grading obtained by lifting the map $X \mapsto -X$ from TM to $\mathbb{C}l(TM)$, with the property that

$$\mathbb{C}l(TM)^\pm E^+ \subset E^\pm, \ \mathbb{C}l(TM)^\pm E^- \subset E^\mp,$$

for any graded Dirac bundle E.

We are now interested in splitting the Dirac operator $D = D_M^E$ along the fibration $\pi : M \to B$ into a "horizontal" and a "vertical" part. The notion of horizontality we use will be introduced below, while we will call a differential operator Q on $C_c^1(M, E)$ *vertical* if Q commutes with multiplication by functions pulled back from the base, i.e. if Q differentiates only in fiber directions; if Q is of first order this is also equivalent to saying that

$$\hat{Q}(\xi) = 0, \ \xi \in T_H^*M, \tag{1.7}$$

with $\hat{Q}$ the principal symbol of Q. The desired splitting of D will reflect the geometry of the fibration π, through the second fundamental form, which is defined for $X, Y \in T_VM$ and $Z \in T_HM$ by

$$\begin{aligned} \langle II(X,Y), Z\rangle &= \langle \nabla_Z X - P_V[Z,X], Y\rangle \\ &= \langle \nabla_X Z, Y\rangle \\ &= -\langle \nabla_X Y, Z\rangle; \end{aligned} \tag{1.8}$$

and through the curvature of π, which is for $Z_1, Z_2 \in T_HM$ defined as

$$R_{Z_1,Z_2} := -P_V[Z_1, Z_2].$$

Before we state the results on the splitting of D we need to introduce some notation concerning local orthonormal frames. We will always denote by $(e_i)_{i=1}^h$ and $(f_j)_{j=1}^v$an oriented local orthonormal frame for T_HM and T_VM, respectively, where $h = \dim B$

and $v := \dim F$ denote the "horizontal" and "vertical" dimensions, with $h + v = m := \dim M$, and we assume that $\{e_1, \ldots, f_v\}$ is oriented in TM. More specifically, we may assume that $(e_i)_{i=1}^h$ consists of the horizontal lifts of an oriented local orthonormal frame $(\underline{e}_i)_{i=1}^h$ for TB; if this frame is defined in some open set U then $(e_i)_{i=1}^h$ is defined in $\pi^{-1}(U)$.

There are two operators generated by D which naturally belong to the horizontal and the vertical space, respectively, to wit

$$\tilde{D}_H := \sum_{i=1}^{h} \mathrm{cl}(e_i)\nabla^E_{e_i}, \tag{1.9}$$

$$\tilde{D}_V := \sum_{j=1}^{v} \mathrm{cl}(f_j)\nabla^E_{f_j}, \tag{1.10}$$

such that $D = \tilde{D}_H + \tilde{D}_V$. However, these operators are not easy to interpret and in spite of having a symmetric principal symbol, they are not symmetric in general. This defect is easily cured as follows. Since D is symmetric on $C_c^1(M, E)$, i. e. $D = D^\dagger$, its formal adjoint, we obtain

$$\begin{aligned} D &= \frac{1}{2}(\tilde{D}_H + \tilde{D}_H^\dagger) + \frac{1}{2}(\tilde{D}_V + \tilde{D}_V^\dagger) \\ &=: D_H + D_V, \end{aligned} \tag{1.11}$$

with $D_{H/V}$ symmetric. But since $\tilde{D}_V$ has symmetric principal symbol, we see that

$$\tilde{D}_V^\dagger = \tilde{D}_V + \beta_1, \tag{1.12}$$

with some endomorphism $\beta_1 \in C^\infty(M, \mathrm{End}\, E)$ such that

$$D_H = \tilde{D}_H - \frac{1}{2}\beta_1, \tag{1.13}$$

$$D_V = \tilde{D}_V + \frac{1}{2}\beta_1; \tag{1.14}$$

note that β_1 is necessarily skew-symmetric.

Lemma 1.1. — 1. *In* (1.12), *we have*

$$\beta_1 = -v\, \mathrm{cl}(H_F), \tag{1.15}$$

where

$$H_F := -\frac{1}{v} \sum_{j=1}^{v} P_H \nabla^{TM}_{f_j} f_j$$

is the mean curvature vector field of the fibers of π.
2. *For any horizontal vector field* Z *on* M *we have*

$$\mathrm{cl}(Z) D_V + D_V\, \mathrm{cl}(Z) = 0. \tag{1.16}$$

Proof. — 1. We compute $\tilde{D}_V^\dagger$ by calculating for $\sigma_k \in C_c^1(M,E)$, $k=1,2$, the expression

$$
\begin{aligned}
(\tilde{D}_V\sigma_1,\sigma_2)_{L^2(M,E)} &- (\sigma_1,\tilde{D}_V\sigma_2)_{L^2(M,E)} \\
&= \sum_{j=1}^{v}\int_M \big(\langle \mathrm{cl}(f_j)\nabla^E_{f_j}\sigma_1,\sigma_2\rangle_E - \langle\sigma_1,\mathrm{cl}(f_j)\nabla^E_{f_j}\sigma_2\rangle_E\big) \\
&= \sum_{j=1}^{v}\int_M \big(-f_j\langle\sigma_1,\mathrm{cl}(f_j)\sigma_2\rangle_E + \langle\sigma_1,\mathrm{cl}(\nabla^{TM}_{f_j}f_j)\sigma_2\rangle_E\big) \\
&= \sum_{j=1}^{v}\int_M \big(-f_j\langle\sigma_1,\mathrm{cl}(f_j)\sigma_2\rangle_E + \langle\sigma_1,\mathrm{cl}(\nabla^{T_VM}_{f_j}f_j)\sigma_2\rangle_E\big) \\
&\quad - (\sigma_1, v\,\mathrm{cl}(H_F)\sigma_2)_{L^2(M,E)},
\end{aligned}
\tag{1.17}
$$

where we have used the properties (1.3) through (1.5). Now we introduce a vertical vector field, X, by setting

$$\langle X,Y\rangle_{T_VM} := \langle\sigma_1,\mathrm{cl}(Y)\sigma_2\rangle_E,\ Y\in C(M,T_VM).$$

Then it is easy to see that the integrand in (1.17) equals the divergence of $X|F_b$ and hence vanishes upon integration over F_b, for any $b\in B$. It follows that

$$\tilde{D}_V^\dagger - \tilde{D}_V = -v\,\mathrm{cl}(H_F),$$

as claimed.

2. We compute, using again the basic relations (1.3) through (1.5),

$$
\begin{aligned}
\mathrm{cl}(X)D_V + D_V\,\mathrm{cl}(X) &= \mathrm{cl}(X)\tilde{D}_V + \tilde{D}_V\,\mathrm{cl}(X) + v\langle X,H_F\rangle_{TM} \\
&= \sum_j \big(\mathrm{cl}(X)\,\mathrm{cl}(f_j)\nabla^E_{f_j} + \mathrm{cl}(f_j)\nabla^E_{f_j}\,\mathrm{cl}(X)\big) + v\langle X,H_F\rangle_{TM} \\
&= \sum_j \mathrm{cl}(f_j)\,\mathrm{cl}(\nabla^{TM}_{f_j}X) + v\langle X,H_F\rangle_{TM} \\
&= \Big(\sum_{j,l}\mathrm{cl}(f_j)\,\mathrm{cl}(f_l)\langle\nabla^{TM}_{f_j}X,f_l\rangle_{TM} + v\langle X,H_F\rangle_{TM}\Big) \\
&= \sum_{j\neq l}\mathrm{cl}(f_j)\,\mathrm{cl}(f_l)\langle X,\nabla^{TM}_{f_j}f_l\rangle_{TM} \\
&= 0.
\end{aligned}
$$

□

We will use below a stronger property of this decomposition, namely that (in the case of D^Λ)

$$\tilde{D}_{HV} := D_HD_V + D_VD_H \tag{1.18}$$

is a first order vertical differential operator. Note that while $\tilde{D}_{HV}$ is always of first order, it need not be vertical in general. But this can be achieved if we further modify the decomposition (1.11) by bringing in the curvature of π.