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1 Introduction and Results

Let $Q = Q(z, \bar{z})$ be a subharmonic and non-harmonic polynomial on the complex plane $\mathbf{C}$ with real values. Then the degree the non-harmonic part Q^N of Q is an even positive number $2k \in \mathbf{N}^*$. In their paper [1], F. Berteloot and G. Cœuré proved that the domain $\Omega_Q = \{(w, z) \in \mathbf{C}^2 \mid \operatorname{Re} w + Q(z, \bar{z}) < 0\}$ is **hyperbolic** for every Q like above. In this note, we consider the positive cone M of all such polynomials and the associated domains $\Omega_Q \subset \mathbf{C}^2$.

Let $Q_1, Q_2 \in M$ and $\Omega_{Q_1}, \Omega_{Q_2}$ be the associated domains. In what follows, we use also Ω, Ω_1, Ω_2 instead of Ω_Q, Ω_{Q_1}, Ω_{Q_2} if there is no confusion possible. First, we introduce an equivalence relation on the cone M.

Definition 1.1 *Let $Q_1, Q_2 \in M$. We say that Q_1 and Q_2 are equivalent $Q_1 \sim Q_2$, if there is a real number $\rho > 0$, a holomorphic polynomial $p(z)$ and an automorphism $g(z)$ of $\mathbf{C}$ such that*

$$Q_1(z, \bar{z}) = \rho \operatorname{Re}(p(z)) + \rho Q_2(g(z), \overline{g(z)}). \tag{1.1}$$

On the other hand, there is another equivalence relation on M given by the biholomorphy of the domains Ω_{Q_1} and Ω_{Q_2}. The first results states that these two equivalence relations are the same.

Theorem 1.2 *Let Q_1, $Q_2 \in M$. Then Ω_1 and Ω_2 are biholomorphic, if and only if the two polynomials Q_1 and Q_2 are equivalent in the sense of definition 1.1. In particular the degrees of the non-harmonic parts Q_1^N and Q_2^N are equal, if the domains Ω_1 and Ω_2 are biholomorphic.*

The fact that Ω is hyperbolic implies that the holomorphic automorphism group $\operatorname{Aut}_{\mathcal{O}}(\Omega)$ is a real Lie group and that all isotropy groups of the action of $\mathrm{Aut}_{\mathcal{O}}(\Omega)$ on Ω are compact [3]. We denote by G, G_1, G_2 the connected identity components of $\mathrm{Aut}_{\mathcal{O}}(\Omega)$, $\mathrm{Aut}_{\mathcal{O}}(\Omega_1)$, $\mathrm{Aut}_{\mathcal{O}}(\Omega_2)$. Clearly, if Ω_1 and Ω_2 are biholomorphic, then G_1 and G_2 are isomorphic.

Let $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ denote the Lie algebras of G, G_1, G_2.

Let J, J_1, J_2 denote the subgroups of G, G_1, G_2 generated by the translation $\{(w, z) \mapsto (w+it, z) \mid t \in \mathbf{R}\}$ and j, j_1, j_2 their Lie algebras. Hence the dimension of G, G_1, G_2 is at least one.

The second result gives a "canonical" defining polynomial for the domain Ω if $\dim_{\mathbf{R}} \mathcal{G} \geq 2$.

Theorem 1.3 *Let* $\Omega = \{\mathrm{Re}\, w + Q(z) < 0\}$ *as above. Assume that* $\dim_{\mathbf{R}} G \geq 2$. *Then there are the following cases :*

a) Ω *is homogeneous. Then* $\Omega \simeq \mathbf{B}_2 = \{|w|^2 + |z|^2 < 1\}$ *and* $Q \sim P_1 \sim P_2$, *where* $P_1(z, \bar{z}) = (\mathrm{Re}\, z)^2$ *and* $P_2(z, \bar{z}) = |z|^2$.

b) Ω *is not homogeneous.*

1) $\dim_{\mathbf{R}} G = 2$. *Then* $\deg Q^N \geq 4$ *and either i)* $Q \sim P_1$ *or ii)* $Q \sim P_2$, *or iii)* $Q \sim P_3$, *where*

i) $P_1(z, \bar{z}) = P_1(\mathrm{Re}\, z)$ *is an element of* M *depending only on* $\mathrm{Re}\, z$ *and* $G \simeq (\mathbf{R}^2, +)$,

ii) $P_2(z, \bar{z}) = P_2(|z|^2)$ *is an element of* M *depending only on* $|z|^2$, *and* $G \simeq \mathbf{R} \times S^1$,

iii) $P_3(z, \bar{z})$ *is a homogeneous polynomial of degree* $2k$, $k \geq 2$, *i.e.* $P_3(\lambda z, \lambda \bar{z}) = \lambda^{2k} P_3(z, \bar{z})$ *for all* $\lambda \in \mathbf{R}$ *and* G *is the non-abelian two dimensional real Lie group.*

2) $\dim_{\mathbf{R}} G \geq 3$. *Then* $\deg Q^N \geq 4$ *and either i)* $Q \sim P_1$ *or ii)* $Q \sim P_2$ *where*

i) $P_1(z, \bar{z}) = (\mathrm{Re}\, z)^{2k}$ *and* G *is 3-dimensional and solvable,*

ii) $P_2(z, \bar{z}) = |z|^{2k}$ *and* G *is 4-dimensional and contains a finite covering of* $SL_2(\mathbf{R})$.

We are going to prove the two theorems simultaneously by distinguishing the dimension of G. First we handle the one and two-dimensional cases, then the homogeneous case and we finish with the three and higher dimensional cases.

Before doing so, we prove the easy direction of theorem1.1.

Lemma 1.4 *If* $Q_1 \sim Q_2$, *then* Ω_1 *and* Ω_2 *are biholomorphic.*

Proof : Assume (1.1). Let $\Psi = (\Psi_1, \Psi_2)$ be the biholomorphic map of $\mathbf{C}^2$ defined by

$$(*) \qquad \begin{cases} \Psi_1(w,z) = \frac{1}{\rho} w + p(z) \\ \Psi_2(w,z) = g(z) \end{cases}$$

Then $\Psi(\Omega_1) = \Omega_2$. ∎

Remark 1.5 *In what follows we will often make a global coordinate change in* $\mathbf{C}^2$ *like* $(*)$*, which is coherent with the equivalence of the defining polynomials. In the following, we take the notation from above.*

2 The one-dimensional case

Let $\Psi : \Omega_1 \to \Omega_2$ be a biholomorphic map. For a subgroup $N \subset G_2$ let $\Psi^*(N)$ be the group $\Psi^{-1} \circ N \circ \Psi \subset G_1$.

Lemma 2.1 *Assume that* $\Psi^*(J_2) = J_1$. *Then* $Q_1 \sim Q_2$.

Proof : From our hypothesis it follows that there is a non-zero real number ρ such that

$$\Psi^{-1} \circ T_t \circ \Psi = T_{\rho t}, \ (T_t(w,z) = (w + it, z)),$$

since Ψ^* is a continuous group isomorphism of two copies of $\mathbf{R}$.

So we get with $\Psi = (\Psi_1, \Psi_2)$

$$\begin{aligned} \Psi_1(w,z) + it &= \Psi_1(w + i\rho t, z) \\ \Psi_2(w,z) &= \Psi_2(w + i\rho t, z) \end{aligned}$$

which implies :

$$\begin{aligned} \Psi_1(w,z) &= \frac{1}{\rho} w + p(z) \\ \Psi_2(w,z) &= g(z) \end{aligned}$$

with $p \in \mathcal{O}(\mathbf{C})$ and $g \in \mathrm{Aut}_{\mathcal{O}}(\mathbf{C})$, since the projection $\pi : \mathbf{C}^2 \to \mathbf{C}$, $(w,z) \mapsto z$ is surjective on Ω_1 and Ω_2.

Therefore Ψ is a biholomorphic map of $\mathbf{C}^2$ which maps Ω_1 to Ω_2 and so we have

$$\begin{aligned} \Omega_1 &= \{\mathrm{Re}\, w + Q_1(z, \bar{z}) < 0\} = \Psi^{-1}(\Omega_2) \\ &= \{\mathrm{Re}(\frac{1}{\rho} w + p(z)) + Q_2(g(z), \overline{g(z)}) < 0\} \\ &= \{\mathrm{Re}\, w + \rho\, \mathrm{Re}\, p(z) + \rho Q_2(g(z), \overline{g(z)}) < 0\}. \end{aligned}$$

It follows that

$$Q_1(z,\bar{z}) = \rho\,\mathrm{Re}\,p(z) + \rho Q_2(g(z),\overline{g(z)}).$$

This equality implies the positivity of ρ and the fact that the holomorphic function $p(z)$ is already a polynomial. Hence $Q_1 \sim Q_2$. ∎

We mention the following direct consequence, which is the statement of theorem 1.2 in the case $\dim_{\mathbf{R}} G_1 = 1$.

Corollary 2.2 *If* $\dim_{\mathbf{R}} G_1 = 1$, *then* Q_1 *and* Q_2 *are equivalent.*

Proof : Here we have $G_1 = J_1$ and $G_2 = J_2$, hence $\Psi^*(J_2) = J_1$. ∎

3 The two-dimensional case

We are going to handle this case in a sequence of lemmas. We always assume that there is a two-dimensional subgroup $H \subset G$ such that $J \subset H$. Since $J \subset G$ is a closed subgroup isomorphic to $\mathbf{R}$ there are two possibilities for H :

i) H is abelian and non-compact.

ii) H is the solvable two dimensional non-abelian Lie group.

Lemma 3.1 *Suppose that* H *is abelian. Then* $Q \sim P_1$ *or* $Q \sim P_2$, *where* $P_1(z,\bar{z}) = P_1(\mathrm{Re}\,z)$ *is an element of* M *which depends only on* $\mathrm{Re}\,z$, *or* $P_2(z,\bar{z}) = P_2(|z|^2)$ *is an element of* M *which depends only on* $|z|^2$.

In the first case, the domain $\{\mathrm{Re}\,w + P_1(\mathrm{Re}\,z) < 0\}$ *realizes the domain* Ω *as a tube domain.*

Proof : Let $L = \{\sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbf{R}\}$ be a one parameter group of H such that L and J generate H. The group H being abelian implies that L and J commute and so we get for all $s,t \in \mathbf{R}$:

$$\begin{aligned}\sigma_1^t(w+is,z) &= \sigma_1^t(w,z) + is\\ \sigma_1^t(w+is,z) &= \sigma_1^t(w,z).\end{aligned}$$

The restriction of the projection $\pi : (w,z) \to z$ from $\mathbf{C}^2$ to Ω being surjective and the second equality imply that

$$\sigma_2^t(w,z) = \sigma_2^t(z)$$