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COMPLEX DYNAMICS IN HIGHER DIMENSION. I.

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1. Introduction

Given a polynomial equation $P(x) = a_n x^n + \cdots + a_0 = 0$, in one variable, x, one asks what are the solutions. The main advantage of the complex number system is that if x is allowed to be complex then the solutions always exist. However, to find the actual values of the solutions is impossible. One can only find approximate solutions.

A traditional method is Newton's method. One starts with a value x_0 and finds inductively a sequence $\{x_n\}$, $x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$. If x_0 is near a simple root, this sequence converges to this root.

Shröder [Sc] was the first to study Newton's method for complex numbers. He was led to the study of iteration of the rational function $R(z) := z - \frac{P(z)}{P'(z)}$. Mainly he studied the local behavior of rational functions near attractive fixed points, $R(z_0) = z_0$, $| R'(z_0) |< 1$. He actually studied general rational functions rather than the special ones from Newton's method, because he discovered that Newton's could be replaced by infinitely many rational functions.

If instead one considers polynomial equation in two (or more) variables, $P(x,y) = Q(x,y) = 0$, where $P(x,y) = \sum a_{n,m} x^n y^n$, one is likewise led to study iteration of rational fonctions in two or more variables. In this case Newton's method takes the inductive form

$$(x_{n+1}, y_{n+1}) = R(x_n, y_n)$$

where the rational map R is given by

$$R(x,y) = (x,y) - \frac{1}{P_x Q_y - P_y Q_x}(PQ_y - QP_y, QP_x - PQ_x).$$

As in one variable there is an infinite family of other rational maps that could be used as well. The simplest one is $R(x,y) = (x,y) - A(P,Q)$ where A is a constant matrix equal to the inverse of the Jacobian matrix of $(P, Q)'$ at some point close to a fixed point.

More precisely consider the mapping in $\mathbf{C}^2$ given by

$$(P, Q) = \left(\frac{1}{2}x - (x - 2y)^2, \frac{1}{2}y - x^2\right).$$

Obviously $(0, 0)$ is a root of the system $P = 0$, $Q = 0$. If we apply the Schroder method to this system with $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ we get in homogeneous coordinate the mapping $f[x : y : t] = [2(z - 2w)^2 : 2z^2 : t^2]$ which is a holomorphic map in $\mathbf{P}^2$. For any invertible matrix B the map $g = I - B(P, Q)$ in homogeneous coordinates is a holomorphic map of $\mathbf{P}^2$.

The analogue of Schröder's study indicated above is the local study of R around attractive fixed points. This was studied extensively in dimension 2, starting by Leau [Le] in the end of the last century and carried through by Lattes [La] and Fatou [Fa].

As far as the global study of iteration is concerned, that is, if we start with a value x_0, perhaps far from the roots of the polynomial, does Newton's method still converge? Schröder was able to decide this only for quadratic polynomials. In this case he found that there is a circle in the sphere, $\mathbf{C} \cup \{\infty\} = \mathbf{P}^1$, dividing it into two open sets. Each of these open sets contains one of the two roots and each starting point x_0 in these open sets give a sequence $\{x_n\}$ by Newton's method converging to the root in the same open set.

The global study of iteration in one variable only became possible in the second decade of this century after the introduction by Montel of normal families, in particular the normality of the family of holomorphic maps from the unit disc to the sphere $\mathbf{P}^1$ minus three points is crucial.

The analogue of this in higher dimensions was unavailable at Fatou's time, so essentially all the study of iteration of rational maps was local.

In this paper we will discuss mainly global questions of iteration of rational maps in higher dimension. The analogue of Montel's Theorem comes from the Kobayashi hyperbolicity of the complement of certain complex hypersurfaces in $\mathbf{P}^k$, the complex projective space of dimension k.

We start, here, with some basic facts on holomorphic endomorphisms of $\mathbf{P}^k$ (i.e. holomorphic maps). For simplicity we sometimes restrict our attention to $\mathbf{P}^2$. In a forthcoming paper we will study the structure of Julia's and Fatou components.

In section 2 we discuss some basic properties of holomorphic and meromorphic maps on $\mathbf{P}^k$.

Section 3 is an estimate of the number of periodic points, counted without multiplicity.

Then in section 4 we give a description of the family of exceptional maps. This family generalizes the map $z \to z^d$ on $\mathbf{P}^1$ which is characterized by the property that the points $\{0, \infty\}$ are totally invariant.

In section 5 we discuss the Kobayashi hyperbolicity of the complement of part of the critical orbit. We show that this holds for a Zariski dense set of maps. See Theorem 5.3 for a precise statement.

In section 6 we consider expansive properties of the map in the complement of the closure of the critical orbit under suitable hyperbolicity assumption and finally, in section 7, we classify critically finite maps in $\mathbf{P}^2$.

2. Holomorphic maps, Fatou and Julia sets.

We first describe the holomorphic maps from $\mathbf{P}^k$ to $\mathbf{P}^k$.

THEOREM 2.1. *Let f be a non constant holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. Then f is given in homogeneous coordinates by $[f_0 : f_1 : \cdots : f_k]$ where each f_j is a homogeneous polynomial of degree d and the f_j have no common zero except the origin.*

Proof. Let $[z_0 : z_1 : \cdots : z_k]$ be homogeneous coordinates in $\mathbf{P}^k$. We can assume that the image of f is not contained in any $(z_j = 0)$ (otherwise rotate coordinates). By the Weierstrass-Hurwitz Theorem [Gu] it follows that each of the meromorphic functions $\frac{z_i}{z_0} \circ f$ is a quotient of two homogeneous polynomials $\frac{F_j}{G_j}$ of the same degree.

Let $\tilde{F}$ denote the map $[\tilde{F}_0 : \cdots : \tilde{F}_k]$ where the $\tilde{F}_j$'s are homogeneous polynomials of the same degree obtained by dividing out common factors from the polynomials $\frac{F_j}{G_j} \cdot \Pi G_\ell$. We will show that $\tilde{F}$ is a lifting of f to $\mathbf{C}^{k+1}$. For this we only need to show that the $\tilde{F}_j$ have no common zeros except the origin. Suppose to the contrary that $p \in \mathbf{C}^{k+1} \mid (0)$ is a common zero. Choose a local lifting $\tilde{f} = [f_0 : \cdots : f_k]$ of f in a neighborhood of p. We may assume that one of the $f_j \equiv 1$. Say $f_0 \equiv 1$. Then it follows that $\tilde{F}_j = \tilde{F}_0 f_j$ and that $\tilde{F}_0(p) = 0$. But this implies that the common zero set of the $\tilde{F}_j$ is a complex hypersurface, which implies that they have a common factor, contradicting that we have already divided out all common factors.

Let $\mathcal{H}$ denote the space of non constant holomorphic maps on $\mathbf{P}^k$ and $\mathcal{H}_d$ the holomorphic maps given by homogeneous polynomials of degree d. Observe that $\mathcal{H}$ is stable under composition.

On the other hand there are the (not necessarily everywhere well defined) maps of degree d from $\mathbf{P}^k$ to $\mathbf{P}^k$, which are given in homogeneous coordinates by $[f_0 : f_1 : \cdots : f_k]$, but now the degree d homogeneous polynomials f_j are allowed to have common zeros. This later space is easily identified with $\mathbf{P}^N$ where $N = (k+1)\frac{(d+k)!}{d!k!} - 1$.

We will also consider the space $\mathcal{M}_d$ of meromorphic maps, consisting of those $[f_0 : \cdots : f_k]$ in P^N which have maximal rank on some nonempty open set.

It follows from Bezout's theorem that for f in $\mathcal{H}_d$ the number of points in $f^{-1}(a)$ is d^k counting multiplicity. Consequently f is of maximal rank and hence $\mathcal{H}_d \subset \mathcal{M}_d \subset \mathbf{P}^N$.

In analogy with one complex variable we define the Fatou set and Julia sets of a holomorphic map f in $\mathcal{H}_d$. More precisely we have the following definition.

DEFINITION 2.2. Given $f : \mathbf{P}^k \to \mathbf{P}^k$ in $\mathcal{H}_d$, $0 \leq \ell \leq k-1$, a point $p \in \mathbf{P}^k$ belongs to the Fatou set $\mathcal{F}_\ell$ if there exists a neighborhood $U(p)$ such that for every $q \in U(p)$ there exists a complex variety X_q through q of codimension ℓ and $\{f^n |_{X_q}\}$ is equicontinuous.

Observe that $\mathcal{F}_0$ is the largest open set where $\{f^n\}$ is equicontinuous. We call $\mathcal{F}_0$ the Fatou set. Also observe that each $\mathcal{F}_\ell$ is open and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{k-1}$.

Correspondingly, let $\mathcal{J}_\ell = \mathbf{P}^k \backslash \mathcal{F}_\ell$. We call $\mathcal{J}_0$ the Julia set.

THEOREM 2.3. *The Julia set of a holomorphic map in*$\mathcal{H}_d$, $d \geq 2$, *is always non empty.*

Proof. Assume $\mathcal{F}_0 = \mathbf{P}^k$. Let h be the limit of a subsequence $\{f^{n_k}\}$. Then h is a non constant holomorphic map of finite degree. As in one variable this contradicts that the degrees of f^{n_k} are unbounded, see [Mi].

THEOREM 2.4. *The sets*$\mathcal{H}_d$ *and*$\mathcal{M}_d$ *are Zariski open sets of*$\mathbf{P}^N$. *In particular*$\mathcal{H}_d$ *and*$\mathcal{M}_d$ *are connected. If* $f \in \mathcal{H}_d$, *then the critical set of* f *is an algebraic variety of degree*$(k+1)(d-1)$.

Proof. Consider $\sum$, the analytic set in $\mathbf{P}^N \times \mathbf{P}^k$ defined by the equation $f(z) = 0$. Let $\sum_d$ be the projection of $\sum$ in $\mathbf{P}^N$. Then $\sum_d$ is equal to $\mathbf{P}^N \backslash \mathcal{H}_d$. Since the projection is proper, by Tarski Theorem, we get that $\sum_d$ is an analytic set. The fact that $\mathcal{M}_d$ is Zariski open follows from the equation $\mathbf{P}^N \backslash \mathcal{M}_d = \bigcap_{z \in \mathbf{P}^k} \{f; J(f,z) = 0\}$ where $J(f,z)$ is the Jacobian of the lifted map on $\mathbf{C}^{k+1}$.

Let $f = [f_0 : f_1 : \cdots : f_k] \in \mathcal{H}_d$. Then the critical set of f is the projection