Astérisque

LAWRENCE BREEN

On the classification of 2-gerbes and 2-stacks

Astérisque, tome 225 (1994)

<http://www.numdam.org/item?id=AST_1994__225__1_0>

225 ASTÉRISQUE

1994

ON THE CLASSIFICATION OF 2-GERBES AND 2-STACKS

Lawrence BREEN

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Classification A.M.S. :

Primaire : 18 G 50
Secondaire : 18 D 05, 18 D 10, 18 D 30, 55 S 45

Contents

0 Introduction

There has in recent years been a great renewal of interest in certain aspects of category theory, due to the realization that this theory provided a powerful tool for dealing with such diverse subjects as knot theory and quantum field theory ([Ca], [Mo-Se]). More recently, this in turn has prompted the search for applications of higher categorical structures. The first such higher categorical structure is that of a 2-category (or a slight variant of it known as a bicategory). To state it briefly, a 2-category $\mathcal{C}$ consists of a set of objects $\mathcal{O}$, and for each pair of objects X, $Y \in \mathcal{O}$, of a category $\mathcal{A}r(X,Y)$ of arrows from X to Y satisfying appropriate axioms. For any positive integer $n \geq 2$, the notion of an n-category is defined iteratively, by attaching to every pair of elements X, Y in the set of objects $\mathcal{O}$ an $(n-1)$-category of arrows. Typical examples of such structures are respectively given by the 2-category of (small) categories, and the 3-category of small 2-categories. Recent applications of the notion of a 2-category include Kapranov and Voevodsky's interpretation [K-V] of the Zamolodchikov tetrahedral equations (which are higher analogs of the well known Yang-Baxter equations), and Fischer's work on higher knot theory [Fi].

The aim of the present text is to investigate the sheaf-theoretical and cohomological structures associated to these higher categories. In the case of ordinary categories, the sheaf-theoretical structure which first comes to mind is the naive notion of a "sheaf of categories" $\mathcal{C}$ on a space X. Such a structure consists of a sheaf of objects $\mathcal{O}$ and a sheaf of arrows $\mathcal{A}$ on X, together with source and target maps $s,t\colon \mathcal{A} \longrightarrow \mathcal{O}$, and an identity map $i\colon \mathcal{O} \longrightarrow \mathcal{A}$ satifying the requisite axioms. This defines, functorially in the open set U of X, a category $\mathcal{C}_U$, whose objects and arrows are respectively the sections of $\mathcal{O}$ and of $\mathcal{A}$ above U. While the given sheaf conditions on $\mathcal{A}$ and $\mathcal{O}$ do provide gluing conditions for both objects and arrows in the categories $\mathcal{C}_U$, the gluing axioms for objects obtained in this manner are too restrictive, and quite unnatural from a category-theory point of view. For this reason, it has proved necessary to introduce the concept of a stack on X. This is defined to be a sheaf of categories endowed with a strengthened gluing axiom for objects. Stacks are fairly familiar, as they play an important role in algebraic geometry, where they provide the most appropriate framework for the theory of moduli spaces ([De-Mu], [L-M]).

The corresponding sheaf-theoretic notions which may be built from 2- and, more generally, from n-categories, (and which are known respectively as 2- and n-stacks) are generally considered to be much more exotic. Their importance was emphasized by Grothendieck in his text which examined the relationship between homotopy theory and topos theory [Gr]. It was also observed by Deligne that an understanding of n-stacks would be necessary if one was to interpret geometrically the higher Chern class terms appearing in the Riemann-Roch formula, along the lines of his discussion in [Del 4] for the terms involving the first Chern class. More recently, Brylinski and McLaughlin [Br-M] have interpreted certain degree 4 characteristic classes for a Lie group in a similar geometric manner, and explored the implications of their construction in conformal field theory. Various sorts of higher level stacks have also appeared elsewhere in the litterature, in a variety of contexts ([Fr], [Ka]).

A drawback in considering n-stacks for $n \geq 3$ is the fact that the presently available explicit definitions of higher level categories are very complicated (we refer to [G-P-S], and [Le] for a discussion of the appropriate axioms in the case of tricategories). No such obstacle exists, however, in the case of 2-stacks. The definition of a 2-category is well understood, and its constituents can readily be represented by diagrams. While we will at times discuss higher stacks, the main aim of the present work is to provide a complete set of cohomological invariants for a 2-stack whose arrows and the 2-arrows are invertible in an appropriate sense. We will refer to such 2-stacks as 2-stacks in 2–groupoids. In the case of ordinary stacks (or 1-stacks), the analogous problem of determining such a set of invariants rapidly reduces to the problem of defining non-abelian degree 2 sheaf cohomology, and of describing in geometric terms those objects which such a cohomology set classifies. These geometric objects were defined by Giraud in [Gi] under the name of gerbes on X, and have been useful in a variety of situations ([De-Mi], [Bry]). At about the same time as these gerbes were being defined by Giraud, Dedecker introduced, mainly in the more restrictive context of group cohomology, certain explicit 2-cocycles with values in a non-abelian group G [Ded 1]. An important feature of Dedecker's theory is the fact that the coefficients of his cohomology theory are not really determined by the groups G themselves, but rather by certain length one complexes of groups $G \longrightarrow \Pi$ satisfying some additional conditions, and which are known as crossed modules. The relationship between the geometric approach to H^2 of Giraud and the cocyclic approach of Dedecker was first discussed, in the abelian case, in [Gi], where it was shown how to associate to a so-called abelian gerbe an ordinary abelian-valued Čech 2-cocycle. In the general (non-abelian) situation, the relation between these two aspects of the theory was worked out in [Br 2], [Br 4]. As in Dedecker's theory, the non-abelian G-valued degree 2 cocycles which are associated to a gerbe on X are to be interpreted as degree 1 cocycles taking their values in appropriate crossed modules.

The question which concerns us here is the corresponding classification problem for 2-stacks. As in the case of 1-stacks, this problem rapidly reduces to a problem in non-abelian cohomology. This consists in defining non-abelian degree 3 cohomology sets, and in determining the geometric objects which these sets classify. We give here a complete solution of this problem in a very general, sheaf theoretic, context. A first attempt at an explicit cocyclic description of a non-abelian H^3 goes back, in a cohomology of groups situation, to [Ded 2] (where the coefficients for the theory were however chosen in an overly restrictive manner). On the geometric side, it had been noticed by Duskin [Du 1] that the geometric objects which degree 3 cohomology classifies are fibered 2-categories satisfying appropriate conditions. We gave a definition of such objects in [Br 3] 4.1, where we called them 2-gerbes. We also observed there that one could associate a particular class of 2-gerbes, which we called the class of $\mathcal{G}$–2-gerbes, to any given stack of monoidal group-like groupoids (or gr–stack) $\mathcal{G}$ on a space X.

While it is possible to give a cohomological description of the full set of equivalence classes of arbitrary 2-gerbes on X, the set of equivalence classes of these $\mathcal{G}$-2-gerbes on X has a particularly pleasant interpretation. Once more, this is to be interpreted as an H^1, but now with values in a somewhat complicated coefficient object. To be a little more precise, let us say that the appropriate coefficient object for such a theory is the "crossed module of gr-stacks" defined by the inner conjugation functor $i: \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G})$ from the given gr–stack $\mathcal{G}$ to the gr–stack $\mathcal{E}q(\mathcal{G})$ of its self-equivalences. More restrictive coefficients for a theory of the non-abelian H^3 are provided by the crossed squares of Loday [Lo], or by the length 2 non-abelian complexes of groups defined by D. Conduché in [Co]. A first illustration of such a theory of the non-abelian H^3 is provided by the problem of classifying extensions of gr–categories and of gr–stacks. This was solved in [Br 3], where it was shown that the classes of extensions of the discrete gr–stack $\underline{K}$ associated to a sheaf of groups K on X by a gr–stack $\mathcal{G}$ on X are classified by the cohomology set $H^1(BK, \mathcal{G} \longrightarrow \mathcal{E}q(\mathcal{G}))$ associated to the classifying space BK of K. It follows from the previous discussion that such extensions therefore correspond to $\mathcal{G}$-2-gerbes on BK. We refer to *op. cit.*, for an explicit description of the non-abelian 3-cocycles associated to such extensions, which generalizes Schreier's well-known description of ordinary group extensions in terms of non-abelian 2-cocycles.

Let us now describe in some detail the contents of the present text. We begin by examining the gluing conditions embodied in the concepts of an n–stack. We then review (§2) the definition of a gerbe on a space X, and give an alternate description of such a gerbe in terms of 2-cocycles. While we already dealt with these questions in [Br 2] and [Br 4] §5, we have found it necessary to return to this topic here, since it paves the way for our subsequent study of 2-gerbes. We have chosen to carry out this discussion in terms of traditional covers of the space X by open sets, instead of working in the more general context of Grothendieck topologies. We hope that this choice of a somewhat more limited framework will make the theory accessible to a wider readership, even though it is with Grothendieck topologies that one often has to deal, both in the context of algebraic geometry and in that of topos theory. Let us however emphasize that the entire discussion carried out here remains valid, without change, in the wider context. Indeed, in order for our results to be directly applicable to the general situation, we have not restricted ourselves, as previous authors, to the Čech cohomology case, a framework which would have been quite adequate (under a paracompactness assumption on X) in dealing with the cohomology of ordinary topological spaces. We work here instead, despite the numerous complications which this entails, with the more general (derived functor) cohomology, which must be described in terms of hypercovers of the space X, rather than in terms of ordinary open covers of X. The "calculus of cocycles" which is then required is already quite intricate at the level of degree 2 cohomology considered in this preliminary section. We have however chosen to work it out in some detail, since we have found this to be quite enlightening. This section ends with three propositions (2.11-2.14) which clarify the relationship between the three related notions of a gerbe, a G–gerbe and an abelian G–gerbe.

Our study of 2-gerbes begins with an examination of the various related conditions by which these 2-gerbes may be defined (§ 3). The two subsequent paragraphs contain the main results of the present work. There we examine, both in the Čech (proposition 4.6 and theorem 5.6) and in the more general hypercover case (proposition 4.10 and theorem 5.9), the