

# *Astérisque*

JEAN-MICHEL BISMUT

**Holomorphic families of immersions and higher  
analytic torsion forms**

*Astérisque*, tome 244 (1997)

<[http://www.numdam.org/item?id=AST\\_1997\\_\\_244\\_\\_R1\\_0](http://www.numdam.org/item?id=AST_1997__244__R1_0)>

© Société mathématique de France, 1997, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

**244**

**ASTÉRISQUE**

**1997**

**HOLOMORPHIC FAMILIES  
OF IMMERSIONS AND HIGHER  
ANALYTIC TORSION FORMS**

**Jean-Michel BISMUT**

**SOCIÉTÉ MATHÉMATIQUE DE FRANCE**

**Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE**

L'auteur remercie l'Institut Universitaire de France (I. U. F.) pour son soutien. Une partie de ce travail a été effectuée à l'Institute for Advanced Study (I. A. S.) de Princeton, de Janvier à Mars 1994, grâce au soutien de la fondation Gould, que l'auteur remercie très vivement. L'auteur remercie également un rapporteur pour ses observations sur ce texte.

# Contents

|                                                                                                        |           |
|--------------------------------------------------------------------------------------------------------|-----------|
| <b>Introduction</b>                                                                                    | <b>1</b>  |
| <b>1 Families of immersions and connections on the relative tangent bundle</b>                         | <b>13</b> |
| 1.1 A canonical connection on the relative tangent bundle of a fibration                               | 13        |
| 1.2 An identity on the connection on the relative tangent bundle . . . .                               | 15        |
| 1.3 Families of immersions and the corresponding connections on the relative tangent bundles . . . . . | 16        |
| <b>2 Kähler fibrations, higher analytic torsion forms and anomaly formulas</b>                         | <b>23</b> |
| 2.1 Kähler fibrations . . . . .                                                                        | 23        |
| 2.2 Complex Hermitian vector spaces and Clifford algebras . . . . .                                    | 26        |
| 2.3 The Levi-Civita superconnection of the fibration . . . . .                                         | 27        |
| 2.4 Superconnection forms and transgression formulas . . . . .                                         | 29        |
| 2.5 The asymptotics of the superconnection forms as $u \rightarrow 0$ . . . . .                        | 30        |
| 2.6 The asymptotics of the superconnection forms as $u \rightarrow +\infty$ . . . . .                  | 31        |
| 2.7 Higher analytic torsion forms . . . . .                                                            | 32        |
| 2.8 Anomaly formulas for the analytic torsion forms . . . . .                                          | 33        |
| <b>3 Kähler fibrations, resolutions, and Bott-Chern currents</b>                                       | <b>35</b> |
| 3.1 A family of double complexes . . . . .                                                             | 35        |
| 3.2 The analytic torsion forms of the double complex . . . . .                                         | 37        |
| 3.3 Assumptions on the metrics on $\xi, \eta$ . . . . .                                                | 40        |
| 3.4 A Bott-Chern current . . . . .                                                                     | 41        |
| <b>4 An identity on two parameters differential forms</b>                                              | <b>43</b> |
| 4.1 A basic identity of differential forms . . . . .                                                   | 43        |
| 4.2 A change of coordinates . . . . .                                                                  | 46        |

|          |                                                                                    |            |
|----------|------------------------------------------------------------------------------------|------------|
| 4.3      | A contour integral . . . . .                                                       | 47         |
| 4.4      | Some elementary identities . . . . .                                               | 48         |
| <b>5</b> | <b>The analytic torsion forms of a short exact sequence</b>                        | <b>51</b>  |
| 5.1      | Short exact sequences and superconnections . . . . .                               | 51         |
| 5.2      | The conjugate superconnections $\mathcal{C}_u$ and $\mathcal{D}_u$ . . . . .       | 54         |
| 5.3      | Generalized supertraces . . . . .                                                  | 55         |
| 5.4      | Transgression formulas and convergence of generalized supertraces .                | 56         |
| 5.5      | Generalized analytic torsion forms . . . . .                                       | 56         |
| 5.6      | Evaluation of the generalized analytic torsion forms . . . . .                     | 57         |
| 5.7      | Equivariant generalized analytic torsion forms . . . . .                           | 58         |
| 5.8      | Some identities on generalized supertraces . . . . .                               | 60         |
| 5.9      | A conjugation formula . . . . .                                                    | 67         |
| <b>6</b> | <b>A proof of Theorem 0.1</b>                                                      | <b>69</b>  |
| 6.1      | The main Theorem . . . . .                                                         | 70         |
| 6.2      | A rescaled metric on $E$ . . . . .                                                 | 71         |
| 6.3      | The left-hand side of (4.26): seven intermediate results . . . . .                 | 72         |
| 6.4      | The asymptotics of the $I_k^0$ 's . . . . .                                        | 74         |
|          | The term $I_1^0$ . . . . .                                                         | 75         |
|          | The term $I_2^0$ . . . . .                                                         | 77         |
|          | The term $I_3^0$ . . . . .                                                         | 79         |
|          | The term $I_4^0$ . . . . .                                                         | 80         |
| 6.5      | The divergences of the left-hand side of (4.26) . . . . .                          | 83         |
| 6.6      | The right-hand side of (4.26): five intermediate results . . . . .                 | 84         |
| 6.7      | The asymptotics of the right-hand side of (4.26) . . . . .                         | 88         |
| 6.8      | Matching the divergences . . . . .                                                 | 105        |
| 6.9      | An identity on Bott-Chern classes and Bott-Chern currents . . . . .                | 107        |
| 6.10     | Proof of Theorem 6.2 . . . . .                                                     | 108        |
| <b>7</b> | <b>A new horizontal bundle on <math>V</math> and the conjugate superconnection</b> | <b>109</b> |
|          | $\tilde{A}_{u,T}$                                                                  |            |
| 7.1      | A formula for $D^X$ and $D^Y$ . . . . .                                            | 110        |
| 7.2      | The canonical exact sequence on $W$ . . . . .                                      | 110        |
| 7.3      | A coordinate system on $V$ near $W$ . . . . .                                      | 111        |
| 7.4      | A splitting of $\xi$ near $W$ . . . . .                                            | 111        |
| 7.5      | A cohomological obstruction to the equality $T^H V _W = T^H W$ . . . .             | 113        |
| 7.6      | An extension of $T^H W$ to $V$ . . . . .                                           | 114        |

|           |                                                                                                                                                            |            |
|-----------|------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| 7.7       | The conjugate superconnection $\tilde{A}_{u,T}$ . . . . .                                                                                                  | 116        |
| 7.8       | A Lichnerowicz formula for $A_{u,T}^2$ and $\tilde{A}_{u,T}^2$ . . . . .                                                                                   | 118        |
| <b>8</b>  | <b>A Taylor expansion of the superconnection <math>\tilde{A}_{1,T}</math> near <math>W</math></b> . . . . .                                                | <b>121</b> |
| 8.1       | A trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ along geodesics normal to $Y$ . . .                                                           | 121        |
| 8.2       | A Taylor expansion for $\tilde{A}_{1,T}$ near $W$ . . . . .                                                                                                | 122        |
| 8.3       | The projection of the superconnection $\mathfrak{B}$ . . . . .                                                                                             | 128        |
| <b>9</b>  | <b>The asymptotics of supertraces involving the operator <math>\exp(-B_{u,T}^2)</math> for large values of <math>u, T</math></b> . . . . .                 | <b>131</b> |
| 9.1       | The spectrum of $B_{u,T}^2$ . . . . .                                                                                                                      | 133        |
| 9.2       | A scaling formula . . . . .                                                                                                                                | 134        |
| 9.3       | Two intermediate results . . . . .                                                                                                                         | 135        |
| 9.4       | A formula for $P^{\xi^-} f^\alpha \nabla_{f_{H,W}^\alpha}^\xi V P^{\xi^-} _W$ and its normal derivative . . .                                              | 137        |
| 9.5       | An embedding of $F$ in $\tilde{E}$ . . . . .                                                                                                               | 138        |
| 9.6       | A Sobolev norm on $E^1$ . . . . .                                                                                                                          | 140        |
| 9.7       | Estimates on the resolvent of $\tilde{A}_T^2$ . . . . .                                                                                                    | 144        |
| 9.8       | Regularizing properties of the resolvent of $\tilde{A}_T^2$ . . . . .                                                                                      | 146        |
| 9.9       | Uniform estimates on the kernel $F_u(\tilde{A}_T^2)$ . . . . .                                                                                             | 150        |
| 9.10      | The matrix structure of $\tilde{A}_T^2$ as $T \rightarrow +\infty$ . . . . .                                                                               | 153        |
| 9.11      | The asymptotics of the operator $F_u(\tilde{A}_T^2)$ as $T \rightarrow +\infty$ . . . . .                                                                  | 155        |
| 9.12      | Proof of Theorem 9.5 . . . . .                                                                                                                             | 155        |
| 9.13      | The operators $\mathcal{G}_{a,b,c,T}$ . . . . .                                                                                                            | 157        |
| 9.14      | Proof of Theorem 9.6 . . . . .                                                                                                                             | 160        |
| 9.15      | Proof of Theorem 6.15 . . . . .                                                                                                                            | 162        |
| 9.16      | Proof of Theorem 6.16 . . . . .                                                                                                                            | 163        |
| <b>10</b> | <b>The asymptotics of the metric <math>g_T^{H(Y,\eta Y)}</math> as <math>T \rightarrow +\infty</math></b> . . . . .                                        | <b>165</b> |
| 10.1      | The lift of sections of $\ker D^Y$ to sections of $\ker A_T^{(0)}$ . . . . .                                                                               | 165        |
| 10.2      | The lift of sections of $\ker D^Y$ to harmonic forms in $E$ for . . . . .                                                                                  | 171        |
| 10.3      | Proof of Theorem 6.10 . . . . .                                                                                                                            | 174        |
| <b>11</b> | <b>The analysis of the two parameter semi-group <math>\exp(-A_{u,T}^2)</math> in the range <math>u \in ]0, 1], T \in [0, \frac{1}{u}]</math></b> . . . . . | <b>177</b> |
| 11.1      | The limit as $u \rightarrow 0$ of $\Phi \operatorname{Tr}_s [N_H \exp(-A_{u,T}^2)]$ . . . . .                                                              | 178        |
| 11.2      | Localization of the problem . . . . .                                                                                                                      | 179        |
| 11.3      | A rescaling of the normal coordinate $Z_0$ . . . . .                                                                                                       | 182        |
| 11.4      | A local coordinate system near $W$ and a trivialization of $\pi_V^* \Lambda(T_R^* S) \hat{\otimes} \Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$ . . . . .       | 182        |

|           |                                                                                                                                                 |            |
|-----------|-------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| 11.5      | The Taylor expansion of the operator $\tilde{B}_{u^2}^{V,2}$                                                                                    | 185        |
| 11.6      | Replacing $X$ by $(T_{\mathbf{R}}X)_{y_0}$                                                                                                      | 186        |
| 11.7      | Rescaling of the variable $Z$ and of the Clifford variables                                                                                     | 187        |
| 11.8      | The matrix structure of $L_{u,T}^{3,Z_0/T}$                                                                                                     | 190        |
| 11.9      | A family of Sobolev spaces with weights                                                                                                         | 192        |
| 11.10     | Proof of Theorem 11.5                                                                                                                           | 193        |
| 11.11     | Proof of Theorem 6.17                                                                                                                           | 193        |
| <b>12</b> | <b>The analysis of the kernel of <math>\tilde{F}_u(A_{u,T/u}^2)</math> for <math>T &gt; 0</math> as <math>u \rightarrow 0</math></b>            | <b>195</b> |
| 12.1      | Localization of the problem                                                                                                                     | 196        |
| 12.2      | A local coordinate system near $y_0 \in W$ and a trivialization of $\Lambda(T^{*(0,1)}X) \hat{\otimes} \xi$                                     | 196        |
| 12.3      | Replacing the fibre $X$ by $(T_{\mathbf{R}}X)_{y_0}$                                                                                            | 196        |
| 12.4      | Rescaling of the variable $Z$ and of the horizontal Clifford variables                                                                          | 197        |
| 12.5      | The asymptotics of the operator $L_{u,T/u}^{3,y_0}$ as $u \rightarrow 0$                                                                        | 198        |
| 12.6      | Proof of Theorem 6.8                                                                                                                            | 203        |
| 12.7      | Proof of the first half of Theorem 6.18                                                                                                         | 203        |
| <b>13</b> | <b>The analysis of the two parameter operator <math>\exp(-A_{u,T}^2)</math> in the range <math>u \in ]0, 1]</math>, <math>T \geq 1/u</math></b> | <b>213</b> |
| 13.1      | A proof of Theorem 6.9: the problem is localizable on $W$                                                                                       | 214        |
| 13.2      | An orthogonal splitting of $TX$ and a connection on $TX$                                                                                        | 216        |
| 13.3      | A local coordinate system near $y_0 \in W$ and a trivialization                                                                                 | 217        |
| 13.4      | Replacing $X$ by $(T_{\mathbf{R}}X)_{y_0}$                                                                                                      | 219        |
| 13.5      | Rescaling of $Z$ and of the horizontal Clifford variables                                                                                       | 220        |
| 13.6      | A formula for $\mathcal{L}_{u,T}^{3,y_0}$                                                                                                       | 222        |
| 13.7      | The algebraic structure of the operator $\mathcal{L}_{u,T}^{3,y_0}$ as $u \rightarrow 0$                                                        | 225        |
| 13.8      | The matrix structure of the operator $\mathcal{L}_{u,T}^{3,y_0}$ as $T \rightarrow +\infty$                                                     | 229        |
| 13.9      | The asymptotics of $F_T k^{1/2} \tilde{A}_{1,T} k^{-1/2} F_T^{-1}$                                                                              | 232        |
| 13.10     | A family of Sobolev spaces with weights                                                                                                         | 235        |
| 13.11     | The operator $\Xi_{y_0}$                                                                                                                        | 236        |
| 13.12     | Proof of Theorem 13.2                                                                                                                           | 237        |
| 13.13     | A proof of Theorem 6.19                                                                                                                         | 237        |
|           | An evaluation of the limit of $\theta_{u,T}$ as $T \rightarrow +\infty$                                                                         | 237        |
|           | The connection ${}^3\nabla \pi \nabla \Lambda(T_{\mathbf{R}}^* S) \hat{\otimes} \Lambda(\mathbf{R}^{2*}) \hat{\otimes} \Lambda(T^{*(0,1)} X)$   | 246        |
|           | The algebraic structure of $\mathcal{L}_{u,T}^{3,y_0}$ as $u \rightarrow 0$                                                                     | 247        |
|           | An estimate on $ \theta_{u,T/u} - \theta_u^* $                                                                                                  | 250        |
| 13.14     | A proof of the second half of Theorem 6.18                                                                                                      | 251        |

|                                                                                                                   |            |
|-------------------------------------------------------------------------------------------------------------------|------------|
| <b>14 A proof of Theorem 0.2</b>                                                                                  | <b>259</b> |
| 14.1 A closed form on $\mathbf{R}_+^* \times \mathbf{R}_+^*$                                                      | 259        |
| 14.2 The asymptotics of the $I_k'^0$ 's                                                                           | 260        |
| The term $I_1'^0$                                                                                                 | 260        |
| The term $I_2'^0$                                                                                                 | 262        |
| The term $I_3'^0$                                                                                                 | 264        |
| The term $I_4'^0$                                                                                                 | 264        |
| The right-hand side of (14.6)                                                                                     | 265        |
| 14.3 Matching the divergences                                                                                     | 266        |
| 14.4 Proof of Theorem 0.2                                                                                         | 266        |
| <b>15 A new derivation of the asymptotics of the generalized supertraces associated to a short exact sequence</b> | <b>267</b> |
| 15.1 A family of immersions                                                                                       | 268        |
| 15.2 The superconnection $\mathcal{B}_T$                                                                          | 269        |
| <b>Bibliography</b>                                                                                               | <b>273</b> |



# Introduction

Let  $i: W \rightarrow V$  be an embedding of smooth complex manifolds. Let  $S$  be a complex manifold. Let  $\pi_V: V \rightarrow S$  be a holomorphic submersion with compact fibre  $X$ , which restricts to a holomorphic submersion  $\pi_W: W \rightarrow S$ , with compact fibre  $Y$ . Then we have the diagram of holomorphic maps

$$(0.1) \quad \begin{array}{ccc} Y & \longrightarrow & W \\ \downarrow i & & \downarrow i \quad \searrow \pi_W \\ X & \longrightarrow & V \xrightarrow{\pi_V} S \end{array}$$

Let  $\eta$  be a holomorphic vector bundle on  $W$ . Let  $(\xi, v)$  be a holomorphic complex of vector bundles on  $V$ , which together with a holomorphic restriction maps  $r: \xi_{0|W} \rightarrow \eta$ , provides a resolution of the sheaf  $i_*\eta$ .

Let  $R\pi_{V*}\xi$ ,  $R\pi_{W*}\eta$  be the direct images of  $\xi, \eta$ . We make the assumption that the  $R^i\pi_{W*}\eta$  are locally free. Then  $R\pi_{V*}\xi$  is also locally free, and moreover we have a canonical isomorphism of  $\mathbf{Z}$ -graded holomorphic vector bundles on  $S$

$$(0.2) \quad R\pi_{V*}\xi \simeq R\pi_{W*}\eta.$$

Also for any  $s \in S$ ,

$$(0.3) \quad \begin{aligned} (R\pi_{V*}\xi)_s &\simeq H(X_s, \xi|_{X_s}), \\ (R\pi_{W*}\eta)_s &\simeq H(Y_s, \eta|_{Y_s}) \end{aligned}$$

(here  $H(X_s, \xi|_{X_s})$  and  $H(Y_s, \eta|_{Y_s})$  denote respectively the hypercohomology of  $\xi|_{X_s}$ , and the cohomology of  $\eta|_{Y_s}$ ).

Let  $\omega^V, \omega^W$  be real (1,1) forms on  $V, W$  which are closed, and which, when restricted to the relative tangent bundles  $TX, TY$ , are the Kähler forms of Hermitian metrics  $g^{TX}, g^{TY}$  on  $TX, TY$ . Let  $g^{\xi_0}, \dots, g^{\xi_m}, g^\eta$  be Hermitian metrics on  $\xi_0, \dots, \xi_m, \eta$ .

Let  $(\Omega(Y, \eta|_Y), \bar{\partial}^Y)$  be the family of relative Dolbeault complexes along the fibres  $Y$ , whose cohomology is equal to  $H(Y, \eta|_Y)$ .