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GEOMETRY OF q-HYPERGEOMETRIC FUNCTIONS, QUANTUM AFFINE ALGEBRAS AND ELLIPTIC QUANTUM GROUPS

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GEOMETRY OF q-HYPERGEOMETRIC FUNCTIONS, QUANTUM AFFINE ALGEBRAS AND ELLIPTIC QUANTUM GROUPS

V. Tarasov, A. Varchenko

Abstract. — The trigonometric quantized Knizhnik-Zamolodchikov (qKZ) equation associated with the quantum group $U_q(\mathfrak{sl}_2)$ is a system of linear difference equations with values in a tensor product of $U_q(\mathfrak{sl}_2)$ Verma modules. We solve the equation in terms of multidimensional q-hypergeometric functions and define a natural isomorphism of the space of solutions and the tensor product of the corresponding evaluation Verma modules over the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ where parameters ρ and γ are related to the parameter q of the quantum group $U_q(\mathfrak{sl}_2)$ and the step p of the qKZ equation via $p = e^{2\pi i\rho}$ and $q = e^{-2\pi i\gamma}$.

We construct asymptotic solutions associated with suitable asymptotic zones and compute the transition functions between the asymptotic solutions in terms of the dynamical elliptic R-matrices. This description of the transition functions gives a connection between representation theories of the quantum loop algebra $U_q(\widetilde{\mathfrak{gl}_2})$ and the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ and is analogous to the Kohno-Drinfeld theorem on the monodromy group of the differential Knizhnik-Zamolodchikov equation.

In order to establish these results we construct a discrete Gauss-Manin connection, in particular, a suitable discrete local system, discrete homology and cohomology groups with coefficients in this local system, and identify an associated difference equation with the qKZ equation.

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Résumé. — L'équation de Knizhnik-Zamolodchikov (qKZ) trigonométrique quantifiée associée au groupe quantique $U_q(\mathfrak{sl}_2)$ est un système linéaire d'équations aux différences finies à valeurs dans un produit tensoriel de $U_q(\mathfrak{sl}_2)$-modules de Verma. Nous résolvons cette équation en terme de fonctions q-hypergéométriques multidimensionnelles et définissons un isomorphisme naturel entre l'espace des solutions et le produit tensoriel des modules de Verma d'évaluation correspondants sur le groupe quantique elliptique $E_{\rho,\gamma}(\mathfrak{sl}_2)$, les paramètres ρ et γ étant reliés aux paramètres q du groupe quantique elliptique $U_q(\mathfrak{sl}_2)$ et p de l'équation qKZ par les relations $p = e^{2\pi i\rho}$ et $q = e^{-2\pi i\gamma}$.

Nous construisons des solutions asymptotiques associées à des secteurs asymptotiques convenables et calculons les fonctions de transition entre les solutions asymptotiques en fonction des R-matrices elliptiques dynamiques. Cette description des fonctions de transition relie la théorie des représentations de l'algèbre de lacets quantique $U_q(\widetilde{\mathfrak{gl}_2})$ à celle du groupe quantique elliptique $E_{\rho,\gamma}(\mathfrak{sl}_2)$ et est analogue au théorème de Kohno-Drinfeld sur le groupe de monodromie de l'équation différentielle de Knizhnik-Zamolodchikov.

Pour établir ces résultats nous construisons une connexion de Gauss-Manin discrète, en particulier un système local discret convenable, des groupes d'homologie et de cohomologie à coefficients dans ce système local, et identifions une équation aux différences associée à ces données à l'équation qKZ.

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1. Introduction

In this paper we solve the trigonometric quantized Knizhnik-Zamolodchikov (qKZ) equation associated with the quantum group $U_q(\mathfrak{sl}_2)$. The trigonometric qKZ equation associated with $U_q(\mathfrak{sl}_2)$ is a system of difference equations for a function $\Psi(z_1,\dots,z_n)$ with values in a tensor product $V_1\otimes\dots\otimes V_n$ of $U_q(\mathfrak{sl}_2)$-modules. The system of equations has the form

$$\begin{aligned}\Psi(z_1,\dots,pz_m,\dots,z_n) &= R_{m,m-1}(pz_m/z_{m-1})\dots R_{m,1}(pz_m/z_1)\,\kappa^{-H_m}\,\times\\ &\times\, R_{m,n}(z_m/z_n)\dots R_{m,m+1}(z_m/z_{m+1})\,\Psi(z_1,\dots,z_n)\,,\end{aligned}$$

$m=1,\dots,n$, where p and κ are parameters of the qKZ equation, H is a generator of the Cartan subalgebra of $U_q(\mathfrak{sl}_2)$, H_m is H acting in the m-th factor, $R_{l,m}(x)$ is the trigonometric R-matrix $R_{V_lV_m}(x)\in\mathrm{End}(V_l\otimes V_m)$ acting in the l-th and m-th factors of the tensor product of $U_q(\mathfrak{sl}_2)$-modules. In this paper we consider only the steps p with absolute value less than 1.

The qKZ equation is an important system of difference equations. The qKZ equations had been introduced in [FR] as equations for matrix elements of vertex operators of the quantum affine algebra. An important special case of the qKZ equation had been introduced earlier in [S] as equations for form factors in massive integrable models of quantum field theory; relevant solutions of these equations had been given therein. Later the qKZ equations were derived as equations for correlation functions in lattice integrable models, cf. [JM] and references therein.

In the quasiclassical limit the qKZ equation turns into the differential Knizhnik-Zamolodchikov equation for conformal blocks of the Wess-Zumino-Witten model of conformal field theory on the sphere.

Asymptotic solutions of the qKZ equation as the step p tends to 1 are closely related to diagonalization of the transfer-matrix of the corresponding lattice integrable model by the algebraic Bethe ansatz method [TV2].

We describe the space of solutions of the qKZ equation in terms of representation theory. Namely, we consider the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ with parameters ρ and γ defined by $p=e^{2\pi i\rho}$, $q=e^{-2\pi i\gamma}$ and the $E_{\rho,\gamma}(\mathfrak{sl}_2)$ modules $V_1^e(z_1),\dots,V_n^e(z_n)$ where $V_m^e(z_m)$ is the evaluation Verma module over $E_{\rho,\gamma}(\mathfrak{sl}_2)$ which corresponds to the $U_q(\mathfrak{sl}_2)$-module V_m. Notice that as a vector space the evaluation Verma module $V_m^e(z_m)$ does not depend on z_m. For every permutation $\tau\in\mathbf{S}^n$ we consider the tensor product $V_{\tau_1}^e\otimes\dots\otimes V_{\tau_n}^e$ and establish a natural isomorphism of the space $\mathbb{S}$ of solutions of the qKZ equation with values in $V_1\otimes\dots\otimes V_n$ and the space $V_{\tau_1}^e\otimes\dots\otimes V_{\tau_n}^e\otimes\mathbb{F}$, where $\mathbb{F}$ is the space of functions of $z_1,\dots,z_n$ which are p-periodic with respect to

each of the variables,

$$\mathbf{C}_\tau : V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes \mathbb{F} \to \mathbb{S},$$

cf. (5.32). Notice that if $\Psi(z)$ is a solution of the qKZ equation and $F(z)$ is a p-periodic function, then also $F(z)\Psi(z)$ is a solution of the qKZ equation.

We call the isomorphisms $\mathbf{C}_\tau$ the tensor coordinates on the space of solutions. The compositions of the isomorphisms are linear maps

$$\mathbf{C}_{\tau,\tau'}(z_1,\ldots,z_n) : V^e_{\tau'_1} \otimes \ldots \otimes V^e_{\tau'_n} \to V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n}$$

depending on $z_1,\ldots,z_n$ and p-periodic with respect to all variables. We call these compositions the transition functions. It turns out that the transition functions are defined in terms of the elliptic R-matrices

$$R^{ell}_{V^e_l V^e_m}(x,\lambda) \in \mathrm{End}\,(V^e_l \otimes V^e_m)$$

acting in tensor products of $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules. Namely, for any permutation τ and for any transposition $(m, m+1)$ the transition function

$$\mathbf{C}_{\tau,\tau\cdot(m,m+1)}(z_1,\ldots,z_n) : V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_{m+1}} \otimes V^e_{\tau_m} \otimes \ldots \otimes V^e_{\tau_n} \to V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n}$$

equals the operator

$$P_{V^e_{\tau_{m+1}} V^e_{\tau_m}} R^{ell}_{V^e_{\tau_{m+1}} V^e_{\tau_m}}\big(z_{\tau_{m+1}}/z_{\tau_m}, (\eta^H \otimes \ldots \otimes \eta^H \otimes \underset{m\text{-th}}{\eta^{-H}} \otimes \ldots \otimes \eta^{-H})\eta^{-1}\kappa\big)$$

twisted by certain adjusting maps, cf. (5.34) and Theorem 4.16. Here $P_{V^e_l V^e_m}$ is the transposition of the tensor factors.

We consider asymptotic zones $|z_{\tau_m}/z_{\tau_{m+1}}| \ll 1$, $m = 1,\ldots,n-1$, labelled by permutations $\tau \in \mathbf{S}^n$. For every asymptotic zone we define a basis of asymptotic solutions of the qKZ equation. We show that for every permutation τ the basis of the corresponding asymptotic solutions is the image of the standard monomial basis in $V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n}$ under the map

$$V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \to V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes 1 \hookrightarrow V^e_{\tau_1} \otimes \ldots \otimes V^e_{\tau_n} \otimes \mathbb{F} \xrightarrow{C_\tau} \mathbb{S},$$

cf. Theorem 6.2. The last two statements express the transition functions between the asymptotic solutions via the elliptic R-matrices.

The trigonometric R-matrix $R_{V_l V_m}(x) \in \mathrm{End}\,(V_l \otimes V_m)$ is defined in terms of the action of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ in the tensor product of $U_q(\mathfrak{sl}_2)$-modules. The quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ is a Hopf algebra which

contains the quantum group $U_q(\mathfrak{sl}_2)$ as a Hopf subalgebra and has a family of homomorphisms $U'_q(\widetilde{\mathfrak{gl}_2}) \to U_q(\mathfrak{sl}_2)$ depending on a parameter. Therefore, each $U_q(\mathfrak{sl}_2)$-module V_m carries a $U'_q(\widetilde{\mathfrak{gl}_2})$-module structure $V_m(x)$ depending on a parameter. For Verma modules V_l, V_m over $U_q(\mathfrak{sl}_2)$ the quantum loop algebra modules $V_l(x) \otimes V_m(y)$ and $V_m(y) \otimes V_l(x)$ are isomorphic for generic x, y. Moreover, for irreducible $U_q(\mathfrak{sl}_2)$-modules V_l, V_m the quantum loop algebra modules $V_l(x) \otimes V_m(y)$ and $V_m(y) \otimes V_l(x)$ are irreducible and isomorphic for generic x, y. The map

$$P_{V_l V_m} R_{V_l V_m}(x/y) : V_l(x) \otimes V_m(y) \to V_m(y) \otimes V_l(x)$$

is the unique suitably normalized intertwiner [T], [CP].

The elliptic R-matrix $R^{ell}_{V_l^e V_m^e}(x, \lambda) \in \mathrm{End}(V_l^e \otimes V_m^e)$ is defined in terms of the action of the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$ in the tensor product of evaluation Verma modules. For evaluation Verma modules $V_l^e(x)$, $V_m^e(y)$ over $E_{\rho,\gamma}(\mathfrak{sl}_2)$, the $E_{\rho,\gamma}(\mathfrak{sl}_2)$-modules $V_l^e(x) \otimes V_m^e(y)$ and $V_m^e(y) \otimes V_l^e(x)$ are isomorphic for generic x, y. The map

$$P_{V_l^e V_m^e} R^{ell}_{V_l^e V_m^e}(x/y, \lambda) : V_l^e(x) \otimes V_m^e(y) \to V_m^e(y) \otimes V_l^e(x)$$

is the unique suitably normalized intertwiner [F], [FV].

Our result on the transition functions between asymptotic solutions together with the indicated construction of R-matrices shows that the qKZ equation establishes a connection between representation theories of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ and the elliptic quantum group $E_{\rho,\gamma}(\mathfrak{sl}_2)$. Our result is analogous to the Kohno-Drinfeld theorem on the monodromy group of the differential Knizhnik-Zamolodchikov equation [K], [D2]. The Kohno-Drinfeld theorem establishes a connection between representation theories of a Lie algebra and the corresponding quantum group, see [D2]. Using the ideas of the Kohno-Drinfeld result it was proved in [KL] that the category of representations of a quantum group is equivalent to a suitably defined fusion category of representations of the corresponding affine Lie algebra. Similarly to the Kazhdan-Lusztig theorem one could expect that our result for the difference qKZ equation could be a base for a Kazhdan-Lusztig type result connecting certain categories of representations of quantum affine algebras and elliptic quantum groups, cf. [KS].

In this paper we consider the trigonometric qKZ equation. There are other types of the qKZ equation: the rational qKZ equation [FR] and the elliptic $qKZB$ equation [F]. Here KZB stands for Knizhnik-Zamolodchikov-Bernard,