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LOCAL TAME LIFTING FOR GL(n)
II: WILDLY RAMIFIED SUPERCUSPIDALS

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Abstract. — Let F be a non-Archimedean local field with finite residue field of characteristic p. An irreducible representation σ of the Weil group $\mathcal{W}_F$ of F is called wildly ramified if $\dim \sigma$ is a power of p and $\sigma \not\cong \chi \otimes \sigma$ for any unramified quasicharacter $\chi \neq 1$ of $\mathcal{W}_F$. We write $\mathcal{G}_m^{\mathrm{wr}}(F)$ for the set of equivalence classes of such representations of dimension p^m. An irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$ is wildly ramified if n is a power of p and $\pi \not\cong \pi \otimes (\chi \circ \det)$ for any unramified quasicharacter $\chi \neq 1$ of $F^\times$. We write $\mathcal{A}_m^{\mathrm{wr}}(F)$ for the set of equivalence classses of such representations of $\mathrm{GL}_{p^m}(F)$. In this paper, we do two things. First, we propose a definition of a base change map $\boldsymbol{l}_{K/F} : \mathcal{A}_m^{\mathrm{wr}}(F) \to \mathcal{A}_m^{\mathrm{wr}}(K)$ for any finite tame extension K/F. The construction is explicit and local, being based on the classification of supercuspidal representations of $\mathrm{GL}_n(F)$ (due to C. Bushnell and P.C. Kutzko) and a partial definition of (non-Galois) tame base change (due to the authors). The results apply to local fields F of positive characteristic. When F has characteristic zero and K/F is cyclic of degree prime to p we show that this map coincides with base change in the sense of Arthur and Clozel. Second, when F has characteristic zero, we construct a canonical bijection $\boldsymbol{\pi}_m^F : \mathcal{G}_m^{\mathrm{wr}}(F) \to \mathcal{A}_m^{\mathrm{wr}}(F)$, for each m. We show that this has many of the properties demanded of a Langlands correspondence.

Recently, M. Harris and R. Taylor have announced a proof of the local Langlands conjecture for $\mathrm{GL}_n(F)$, using a global geometric method. This implies the existence of a canonical bijection $\mathcal{L}_m : \mathcal{G}_m^{\mathrm{wr}}(F) \to \mathcal{A}_m^{\mathrm{wr}}(F)$. If $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, there is an unramified quasicharacter χ_σ of $\mathcal{W}_F$ of finite order dividing p^m such that $\boldsymbol{\pi}_m(\sigma) = \mathcal{L}_m(\sigma \otimes \chi_\sigma)$.

We expect that the methods of this paper will lead to another proof of the local Langlands conjecture for GL_n.

***Résumé* (Changement de base local modéré pour $\mathrm{GL}(n)$ II : représentations supercuspidales sauvages)**

Soit F un corps local non archimédien à corps résiduel fini de caractéristique p. Une représentation irréductible σ du groupe de Weil $\mathcal{W}_F$ de F est dite sauvagement ramifiée si $\dim\sigma$ est une puissance de p et $\sigma \not\cong \chi\otimes\sigma$ pour tout quasicaractère non ramifié $\chi \neq 1$ de $\mathcal{W}_F$. Notons $\mathcal{G}_m^{\mathrm{wr}}(F)$ l'ensemble des classes d'isomorphie de telles représentations de dimension p^m. Une représentation irréductible supercuspidale π de $\mathrm{GL}_n(F)$ est dite sauvagement ramifiée si n est une puissance de p et $\pi \not\cong \pi\otimes(\chi\circ\det)$ pour tout quasicaractère non ramifié $\chi\neq 1$ de $F^\times$. Notons $\boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ l'ensemble des classes d'isomorphie de telles représentations de $\mathrm{GL}_{p^m}(F)$. Dans cet article, nous faisons deux choses. En premier, nous proposons une définition d'une application de changement de base $\boldsymbol{l}_{K/F} : \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F) \to \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$, où K/F est une extension finie modérée. La méthode est locale et explicite, basée sur la classification des représentations supercuspidales due à C. Bushnell et Ph. Kutzko et une définition partielle du changement de base modéré (non galoisien), due aux auteurs. Les arguments s'étendent à des corps locaux de caractéristique non nulle. Si le corps F est de caractéristique nulle et que K/F est cyclique de degré premier à p, nous montrons que cette application coïncide avec le changement de base au sens de J. Arthur et L. Clozel. Deuxièmement, dans le cas où F est de caractéristique nulle, nous construisons une bijection canonique $\boldsymbol{\pi}_m^F : \mathcal{G}_m^{\mathrm{wr}}(F) \to \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ qui possède beaucoup des propriétés exigées d'une correspondance de Langlands.

Récemment, M. Harris et R. Taylor ont annoncé une preuve, par voie globale et géométrique, des conjectures de Langlands pour $\mathrm{GL}_n(F)$. Leurs résultats impliquent l'existence d'une bijection canonique $\mathcal{L}_m : \mathcal{G}_m^{\mathrm{wr}}(F) \to \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$. Pour $\sigma\in\mathcal{G}_m^{\mathrm{wr}}(F)$, il existe un quasicaractère non ramifié χ_σ de $\mathcal{W}_F$, d'ordre fini divisant p^m, tel que $\boldsymbol{\pi}_m(\sigma) = \mathcal{L}_m(\sigma\otimes\chi_\sigma)$.

Nous espérons que les méthodes du présent article mèneront à une preuve alternative des conjectures locales de Langlands pour GL_n.

Contents

INTRODUCTION

1. Let F denote a non-Archimedean local field with finite residue field of characteristic p. For the time being, we assume that F has characteristic zero, and so is a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{F}/F$ of F and let $\mathcal{W}_F$ denote the Weil group of $\overline{F}/F$.

For an integer $n \geqslant 1$, let $\mathcal{G}_n(F)$ denote the set of equivalence classes of irreducible continuous (complex) representations of $\mathcal{W}_F$ of dimension n, and $\mathcal{A}_n(F)$ the set of equivalence classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. The local Langlands conjecture for GL_n **[30]** predicts, for each n, the existence of a canonical bijection

$$\boldsymbol{\lambda}_n : \mathcal{G}_n(F) \xrightarrow{\ \approx\ } \mathcal{A}_n(F)$$

satisfying an extensive list of properties (see also **[25]**). In particular, $\boldsymbol{\lambda}_1$ is the bijection implied by local class field theory.

2. We are here concerned with the problem of constructing $\boldsymbol{\lambda}_n$ in a special, but crucial and rather subtle, case.

For an integer $m \geqslant 0$, let $\mathcal{G}_m^{\mathrm{wr}}(F)$ denote the set of $\sigma \in \mathcal{G}_{p^m}(F)$ which remain irreducible on restriction to the wild inertia subgroup of $\mathcal{W}_F$. (Equivalently, $\boldsymbol{\sigma}$ has dimension p^m and $\sigma \not\cong \sigma \otimes \chi$ for any unramified quasicharacter $\chi \neq 1$ of $\mathcal{W}_F$.) From the point of view of Galois theory, the set $\bigcup_m \mathcal{G}_m^{\mathrm{wr}}(F)$ contains the "difficult" representations of $\mathcal{W}_F$, including the primitive ones.

On the other hand, let $\mathcal{A}_m^{\mathrm{wr}}(F)$ denote the set of $\pi \in \mathcal{A}_{p^m}(F)$ with the property that π is not equivalent to the representation $\chi\pi : g \mapsto \chi(\det g)\pi(g)$ for any unramified quasicharacter $\chi \neq 1$ of $F^\times$.

The aim of this paper is to produce, for each $m \geqslant 0$ and each finite field extension $F/\mathbb{Q}_p$, a canonical bijection

$$\boldsymbol{\pi}_m : \mathcal{G}_m^{\mathrm{wr}}(F) \xrightarrow{\approx} \mathcal{A}_m^{\mathrm{wr}}(F).$$

The bijection we construct exhibits many of the properties demanded of a Langlands correspondence $\boldsymbol{\lambda}_{p^m}$. In particular, when $m = 0$, the map $\boldsymbol{\pi}_0$ is that given by class field theory. For $m \geqslant 0$, $\boldsymbol{\pi}_m$ is natural with respect to topological isomorphisms of the base field. It respects contragredience and takes determinants to central quasicharacters. It is compatible with twisting by quasicharacters. Its deepest properties concern local constants: for $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, the Deligne-Langlands local constant $\boldsymbol{\varepsilon}(\sigma, s, \psi_F)$ **[39]** equals the Godement-Jacquet local constant $\boldsymbol{\varepsilon}(\boldsymbol{\pi}_m(\sigma), s, \psi_F)$ **[15]**. (Here, ψ_F is a non-trivial continuous character of the additive group of F.)

3. Since the completion of the original version of this paper (November 1997), there has been considerable progress in this area. This stems from **[16]** in which, following ideas of Drinfeld and Carayol, Harris produces a canonical map $\boldsymbol{\sigma}_n$ from $\mathcal{A}_n(F)$ to the set of equivalence classes of *semisimple* n-dimensional representations of $\mathcal{W}_F$. An argument in **[6]** shows that, for each n, $\boldsymbol{\sigma}_n$ is in fact a bijection $\mathcal{A}_n(F) \to \mathcal{G}_n(F)$. Let us set $\boldsymbol{\lambda}_n = \boldsymbol{\sigma}_n^{-1}$. In **[18]**, it is shown that the family $\{\boldsymbol{\lambda}_n\}$ has all the properties required of a Langlands correspondence. In particular, it preserves local constants of pairs:

$$\boldsymbol{\varepsilon}(\sigma_1 \otimes \sigma_2, s, \psi_F) = \boldsymbol{\varepsilon}(\pi_1 \times \pi_2, s, \psi_F),$$

for $\sigma_i \in \mathcal{G}_{n_i}(F)$ and $\pi_i = \boldsymbol{\lambda}_{n_i}(\sigma_i)$. Here, the second $\boldsymbol{\varepsilon}$ is the local constant of **[24]**, **[35]**. (This property, in the case $p \nmid n_1 n_2$, was established earlier in **[17]**.) However, the construction of $\boldsymbol{\sigma}_n$ in **[16]** is geometric and makes extensive use of global constructions. It gives no information whatsoever about the nature of the correspondence, especially the way it interacts with the structure theory of supercuspidals in **[9]**. The rôle of the present paper has thus become to make the correspondences $\boldsymbol{\lambda}_n$ more explicit, at least in the important case to hand.

A critical question in this regard is therefore whether our family of maps $\{\boldsymbol{\pi}_m\}$ preserves local constants of pairs. We do not answer that question here, but we do show that it preserves *conductors* of pairs: if $\sigma_i \in \mathcal{G}_{m_i}^{\mathrm{wr}}(F)$ and $\pi_i = \boldsymbol{\pi}_{m_i}(\sigma_i)$, $i = 1, 2$, then the exponent $f(\sigma_1 \otimes \sigma_2)$ of the Artin conductor of the tensor product $\sigma_1 \otimes \sigma_2$ is equal to the conductor $f(\pi_1 \times \pi_2)$ of the pair (π_1, π_2).

From results here and properties of the $\{\boldsymbol{\lambda}_n\}$ given in **[6]**, it is straightforward to show that, for $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, *there is an unramified quasicharacter* χ_σ *of* $F^\times$

such that $\boldsymbol{\lambda}_{p^m}(\sigma) \cong \chi_\sigma \boldsymbol{\pi}_m(\sigma)$. *Moreover,* χ_σ *has finite order, strictly dividing* p^m. In particular, $\boldsymbol{\lambda}_p = \boldsymbol{\pi}_1$ on $\mathcal{G}_1^{\mathrm{wr}}(F)$.

We will return elsewhere to the exact relation between $\boldsymbol{\pi}_m$ and $\boldsymbol{\lambda}_{p^m}$.

4. Our approach is based on the fact that the representations in $\mathcal{G}_m^{\mathrm{wr}}(F)$ exhibit a fairly uniform structure. We proceed by uncovering similar structures in $\mathcal{A}_m^{\mathrm{wr}}(F)$ and constructing the map $\boldsymbol{\pi}_m$ to preserve these. First, $\mathcal{G}_m^{\mathrm{wr}}(F)$ has a canonical subset $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ as follows: a representation $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$ lies in $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ if and only if there is a tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_m$$

with each F_{i+1}/F_i cyclic and totally ramified of degree p, and a quasicharacter χ of $F_m^\times$, such that σ is induced from the representation of $\mathcal{W}_{F_m}$ afforded by χ.

There is an analogous subset $\mathcal{A}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ of $\mathcal{A}_m^{\mathrm{wr}}(F)$: a representation π lies in this set if and only if there is a tower of fields $F = L_0 \subset L_1 \subset \cdots \subset L_m$ and a quasicharacter ξ of $L_m^\times$, with each L_{i+1}/L_i cyclic and totally ramified of degree p, such that

$$\pi = \boldsymbol{i}_{L_1/F} \circ \boldsymbol{i}_{L_2/L_1} \circ \cdots \circ \boldsymbol{i}_{L_m/L_{m-1}}(\xi).$$

Here, $\boldsymbol{i}$ denotes the operation of *automorphic induction,* as in **[22]**. One knows (**[20]**, **[6]** 3.8) that, in the notation above,

$$\sigma = \mathrm{Ind}_{\mathcal{W}_{F_m}}^{\mathcal{W}_F}(\chi) \longmapsto \boldsymbol{i}_{F_1/F} \circ \boldsymbol{i}_{F_2/F_1} \circ \cdots \circ \boldsymbol{i}_{F_m/F_{m-1}}(\chi)$$

induces a bijection ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m$ between $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ and $\mathcal{A}\boldsymbol{c}_m^{\mathrm{wr}}(F)$. The maps ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m$ exhibit a multitude of desirable properties; in particular, they *preserve local constants of pairs.* Our bijection $\boldsymbol{\pi}_m$ is to be an extension of ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m$:

$$\boldsymbol{\pi}_m(\sigma) = {}^{\boldsymbol{c}}\boldsymbol{\pi}_m(\sigma), \quad \sigma \in \mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F).$$

5. In general, given a representation $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, there is a finite, tamely ramified, field extension K/F such that the restriction $\sigma_{K/F}$ of σ to $\mathcal{W}_K$ lies in $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$. There is a canonical choice of the extension K/F (up to isomorphism), such that the degree $[K{:}F]$ is prime to p. We specify $\boldsymbol{\pi}_m(\sigma)$ in terms of K/F and the representation ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m(\sigma_{K/F})$.

If the tame extension K/F is *cyclic,* there is an operation on the other side analogous to the restriction process $\sigma \mapsto \sigma_{K/F}$. *Base change,* in the sense of **[1]**, gives a map

$$\boldsymbol{b}_{K/F} : \mathcal{A}_m^{\mathrm{wr}}(F) \longrightarrow \mathcal{A}_m^{\mathrm{wr}}(K);$$

one can easily extend the definition of $\boldsymbol{b}_{K/F}$ to the case where the tame extension K/F is Galois (as in [**5**] 16.5). If the tame extension K/F attached to $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$ is Galois, then global considerations demand that the representation $\pi = \boldsymbol{\pi}_m(\sigma)$ satisfy ${}^{\mathrm{c}}\boldsymbol{\pi}(\sigma_{K/F}) = \boldsymbol{b}_{K/F}(\pi)$. Also, the central quasicharacter of π must correspond to $\det \sigma$ via class field theory. These two conditions determine π uniquely. The real problem is that the extension K/F given by σ will *not,* in general, be Galois. We thus need to define a suitable operation

$$\boldsymbol{l}_{K/F} : \mathcal{A}_m^{\mathrm{wr}}(F) \longrightarrow \mathcal{A}_m^{\mathrm{wr}}(K),$$

for tame extensions K/F of degree prime to p, which generalizes base change. *The explicit construction of the map $\boldsymbol{l}_{K/F}$ is the main point of the paper.*

6. In fact, we shall define the algebraic tame lifting map $\boldsymbol{l}_{K/F}$ for an arbitrary finite tame extension K of any non-Archimedean local field F, characteristic zero or not.

The map $\boldsymbol{l}_{K/F}$ is transitive with respect to the field extension K/F and natural with respect to topological isomorphisms of K. It respects contragredience and twisting with quasicharacters. It "preserves" central quasicharacters, Godement-Jacquet local constants and conductors of pairs, in the sense that its effect on these objects is precisely that predicted by the Langlands conjectures. We give a complete account of the image and the fibres of $\boldsymbol{l}_{K/F}$. We further show that, for K/F cyclic of p-prime degree and F of characteristic 0, we have $\boldsymbol{l}_{K/F}(\pi) = \boldsymbol{b}_{K/F}(\pi)$ for every $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$ and every $m \geqslant 0$. This refines some of the more general results of [**5**], and gives a complete local algebraic description of base change in these circumstances.

A full list of those properties of $\boldsymbol{l}_{K/F}$ needed for this paper is given in §1. The proofs of these occupy §§3–10.

7. Once we have these properties of $\boldsymbol{l}_{K/F}$, the construction of $\boldsymbol{\pi}_m$ is fairly easy. We take $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$ with associated canonical tame extension K/F as above; thus $\sigma_{K/F}$ lies in $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(K)$ and $p \nmid [K{:}F]$. The representation $\pi_{K/F} = {}^{\mathrm{c}}\boldsymbol{\pi}_m(\sigma_{K/F})$ is defined. There is a unique $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$ such that $\boldsymbol{l}_{K/F}(\pi) = \pi_{K/F}$ and whose central quasicharacter corresponds to $\det \sigma$ via class field theory. We put $\pi = \boldsymbol{\pi}_m(\sigma)$.

The details of the construction of $\boldsymbol{\pi}_m$, and the deduction of its properties from those of $\boldsymbol{l}_{K/F}$, are all contained in §2.

We note in passing that, using the maps $\boldsymbol{\pi}_m$ and automorphic induction, it is an easy matter to produce a canonical bijection $\mathcal{G}_{p^n}(F) \to \mathcal{A}_{p^n}(F)$, $n \geqslant 0$.

8. It is a consequence of our constructions that, for given $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$, there exists an extension K/F of degree prime to p such that $\boldsymbol{l}_{K/F}(\pi) \in \mathcal{A}\boldsymbol{c}_m^{\mathrm{wr}}(K)$. This is not easy to prove directly: see **[27]**, **[29]** for a very detailed analysis of the case $p^m = 2$.

One knows already from **[21]** (see also the discussion in **[6]** 3.2) that there is some tame *Galois* extension K'/F such that $\boldsymbol{b}_{K'/F}(\pi) \in \mathcal{A}\boldsymbol{c}_m^{\mathrm{wr}}(K')$. This weaker result underlies everything we do in §2, and also much of **[16]**. Indeed, the proof **[6]** that Harris's map $\boldsymbol{\sigma}_n$ (as in paragraph 3 above) gives a bijection $\mathcal{A}_n(F) \to \mathcal{G}_n(F)$ depends crucially on it.

9. As noted above in paragraph 6, our construction of $\boldsymbol{l}_{K/F}$ is purely algebraic and works equally well in positive characteristic. Unfortunately one cannot use it to produce an explicit version of the characteristic p Langlands correspondences of **[31]** (the construction of which, we note, is again geometric in nature). The reason is simply that we start from the map ${}^{c}\boldsymbol{\pi}_m$. This relies for its definition on base change (or automorphic induction) and base change is not available in positive characteristic. (There is detailed discussion of such matters in **[5]**.)

Be that as it may, our construction of $\boldsymbol{\pi}$ makes no use of base change beyond the definition and basic properties of ${}^{c}\boldsymbol{\pi}$.

Also, while our construction of $\boldsymbol{\pi}$ from ${}^{c}\boldsymbol{\pi}$ can justly claim to be quite explicit, we say nothing concerning ${}^{c}\boldsymbol{\pi}$ itself. This seems to be quite a difficult problem: again see **[27]** for a detailed examination of the case $p^m = 2$, and **[33]** for the case $m = 1$, $p \neq 2$. The explicit conductor formulæ of **[7]** might yield further information in more general cases.

10. We now review the arrangement and more technical aspects of the paper. Our definition of $\boldsymbol{l}_{K/F}$ is necessarily quite novel. Global methods, of the sort used in **[1]**, **[22]**, yield no clues as to how to handle non-Galois extensions K/F. We therefore rely on the local methods of **[9]** and **[5]** to generalize the approach of **[26]**, **[28]**. These methods have the incidental advantage of being characteristic-independent.

Let us take $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$, for some $m \geqslant 1$. The main results of **[9]** give a canonical presentation of π_F as an induced representation, obtained as follows. We recall that a simple stratum $[\mathfrak{A}_F, n_F, 0, \beta]$ (**[9]** 1.5 or "Preliminaries" below) in $A_F = \mathbb{M}_{p^m}(F)$ defines a pair $H^1(\beta, \mathfrak{A}_F) \subset J^1(\beta, \mathfrak{A}_F)$ of compact open subgroups of $G_F = \mathrm{GL}_{p^m}(F)$ and a distinguished finite set $\mathcal{C}(\mathfrak{A}_F, \beta)$ of