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Abstract. — Suppose that $R \subset S$ are regular local rings of a common dimension, which are essentially of finite type over a field k of characteristic zero, such that the quotient field K of S is finite over the quotient field of R. If V is a valuation ring of K which dominates S, then we show that there are sequences of monoidal transforms (blowups of regular primes) $R \to R_1$ and $S \to S_1$ along V such that $R_1 \to S_1$ is a monomial mapping. It follows that a generically finite morphism of nonsingular varieties can be made to be a monomial mapping along a valuation, after blowups of nonsingular subvarieties. We give applications to factorization of birational morphisms and simultaneous resolution of singularities.

Résumé **(Monomialisation et factorisation locales des morphismes)**

Soient $R \subset S$ deux anneaux locaux réguliers de même dimension, essentiellement de type fini sur un corps k de caractéristique zéro, et tels que le corps des fractions K de S est fini sur celui de R. Si V est un anneau de valuation de K dominant S, nous montrons qu'il existe des suites de transformés monoïdaux (éclatements d'idéaux premiers réguliers) $R \to R_1$ et $S \to S_1$ le long de V tels que $R_1 \to S_1$ est une application monomiale. Il s'ensuit qu'un morphisme génériquement fini de variétés non singulières peut être rendu monomial le long d'une valuation après éclatement de sous-variétés non singulières. Nous donnons des applications à la factorisation des morphismes birationnels et à la résolution simultanée des singularités.

Contents

CHAPTER 1

INTRODUCTION

1.1. Statement of the main results

Suppose that we are given a system of equations

$$
\begin{aligned}
x_1 &= f_1(y_1, y_2, \ldots, y_n) \\
&\vdots \\
x_n &= f_n(y_1, y_2, \ldots, y_n)
\end{aligned}
\tag{1}
$$

which is nondegenerate, in the sense that the Jacobian determinate of the system is not (identically) zero. This system is well understood in the special case that $f_1, \ldots, f_n$ are monomials in the variables $y_1, \ldots, y_n$. For instance, by inverting the matrix A of coefficients of the monomials, we can express $y_1, \ldots, y_n$ as rational functions of d-th roots of the variables $x_1, \ldots, x_n$, where d is the determinant of A.

Our main result shows that all solutions of a system (1) can be expressed in the following simple form. There are finitely many charts obtained from a composition of monoidal tranforms in the variables x and y

$$
\begin{aligned}
x_i &= \Phi_i(\overline{x}_1, \ldots, \overline{x}_n), \quad 1 \le i \le n \\
y_i &= \Psi_i(\overline{y}_1, \ldots, \overline{y}_n), \quad 1 \le i \le n
\end{aligned}
$$

such that the transform of the system (1) becomes a system of monomial equations

$$
\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{a_{11}} \cdots \overline{y}_n^{a_{1n}} \\
&\vdots \\
\overline{x}_n &= \overline{y}_1^{a_{n1}} \cdots \overline{y}_n^{a_{nn}}
\end{aligned}
$$

with $\det(a_{ij}) \neq 0$. A monoidal transform is a composition of

(1) a change of variable

(2) a transform

$$x_1 = x_1(1)x_2(1)$$
$$x_i = x_i(1) \qquad \text{if } i > 1.$$

Our solution is constructive, as it consists of a series of algorithms.

This result can be interpreted geometrically as follows. Suppose that $\phi : X \to Y$ is a generically finite morphism of varieties. Then it is possible to construct a finite number of charts X_i and Y_i such that $X_i \to Y_i$ are monomial mappings, the mappings $X_i \to X$ and $Y_i \to Y$ are sequences of blowups of nonsingular subvarieties, and X_i and Y_i form complete systems, in the sense that they can be patched to obtain schemes which satisfy the existence part of the valuative criteria of properness.

Our main result is stated precisely in Theorem 1.1.

***Theorem 1.1* (Monomialization).** — *Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R.*

Let V be a valuation ring of K which dominates S. Then there exist sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R' and there are regular parameters $(x_1, \ldots, x_n)$ in R', $(y_1, \ldots, y_n)$ in S', units $\delta_1, \ldots, \delta_n \in S'$ and a matrix (a_{ij}) of nonnegative integers such that $\det(a_{ij}) \neq 0$ *and*

$$\begin{aligned} x_1 &= y_1^{a_{11}} \cdots y_n^{a_{1n}} \delta_1 \\ &\vdots \\ x_n &= y_1^{a_{n1}} \cdots y_n^{a_{nn}} \delta_n. \end{aligned} \tag{2}$$

With the assumptions of Theorem 1.1, An example of Abhyankar (Theorem 12 **[6]**) shows that it is in general not possible to perform monoidal transforms along V in R and S to obtain $R' \to S'$ such that $R' \to S'$ is (a localization of) a finite map. As such, Theorem 1.1 is the strongest possible local result for generically finite maps.

A more geometric statement of Theorem 1.1 is given in Theorem 1.2. A complete variety over a field k is an integral finite type k-scheme which satisfies the existence part of the valuative criterion for properness (cf. Chapter 8). Complete and separated is equivalent to proper. A toroidal morphism is locally a monomial mapping in uniformizing parameters on an appropriate etale extension.

Theorem 1.2. — *Let k be a field of characteristic zero, $\Phi : X \to Y$ a generically finite morphism of nonsingular proper k-varieties. Then there are birational morphisms of nonsingular complete k-varieties $\alpha : X_1 \to X$ and $\beta : Y_1 \to Y$, and a toroidal morphism $\Psi : X_1 \to Y_1$ such that the diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{\Psi} & Y_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Phi} & Y \end{array}$$

commutes and α and β are locally products of blowups of nonsingular subvarieties. That is, for every $z \in X_1$, there exist affine neighborhoods V_1 of z, V of $x = \alpha(z)$, such that $\alpha : V_1 \to V$ is a finite product of monoidal transforms, and there exist affine neighborhoods W_1 of $\Psi(z)$, W of $y = \alpha(\Psi(z))$, such that $\beta : W_1 \to W$ is a finite product of monoidal transforms.

Here a monoidal transform of a nonsingular k-scheme S is the map $T \to S$ induced by an open subset T of $\mathrm{Proj}(\oplus\mathcal{I}^n)$, where $\mathcal{I}$ is the ideal sheaf of a nonsingular subvariety of S. We give a proof of Theorem 1.2 in chapter 8.

In the special case of dimension two, we can strengthen the conclusions of Theorem 1.2.

Theorem 1.3. — *Let k be a field of characteristic zero, $\Phi : S \to T$ a generically finite morphism of nonsingular proper k-surfaces. Then there are products of blowups of points (quadratic transforms) $\alpha : S_1 \to S$ and $\beta : T_1 \to T$, and a morphism $\Psi : S_1 \to T_1$ such that the diagram*

$$\begin{array}{ccc} S_1 & \xrightarrow{\Psi} & T_1 \\ \downarrow & & \downarrow \\ S & \xrightarrow{\Phi} & T \end{array}$$

commutes, and Ψ is a toroidal morphism.

In the case of complex surfaces, a proof of 1.3 follows from results of Akbulut and King (Chapter 7 of **[8]**).

We also prove, as a corollary of Theorem 1.1, a local theorem on simultaneous resolution of singularities, which is valid in all dimensions. This theorem is proven in dimension 2 (and in all characteristics) by Abhyankar in Theorem 4.8 of his book "Ramification theoretic methods in algebraic geometry" **[3]**.

Theorem 1.4 **(Theorem 1.1 [14])**. — *Let k be a field of characteristic zero, L/k an algebraic function field, K a finite algebraic extension of L, ν a valuation of K/k, and (R, M) a regular local ring with quotient field K, essentially of finite type over k, such that ν dominates R. Then for some sequence of monodial transforms $R \to R^*$ along ν, there exists a normal local ring S^* with quotient field L, essentially of finite type over k, such that R^* is the localization of the integral closure T of S^* in K at a maximal ideal of T.*

Stronger results hold for birational morphisms, morphisms which are an isomorphism on an open set. A birational morphism of nonsingular projective surfaces can be factored by a product of quadratic transforms. This was proved by Zariski, over an algebraically closed field of arbitrary characteristic, as a corollary to a local theorem on factorization (on page 589 of **[37]** and in section II.1 of **[38]**). The most general