Astérisque

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Astérisque, tome 273 (2001)

<http://www.numdam.org/item?id=AST_2001__273__R1_0>

ASTÉRISQUE 273

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Société Mathématique de France 2001
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification.* — Primary 22E46, 32M15, 14L30, 58J70.

Key words and phrases. — Hermitian symmetric space, highest weight representation, nilpotent orbit, associated variety, generalized Whittaker model, multiplicity-free action, dual pair correspondence.

NILPOTENT ORBITS, ASSOCIATED CYCLES AND WHITTAKER MODELS FOR HIGHEST WEIGHT REPRESENTATIONS

Kyo Nishiyama, Hiroyuki Ochiai, Kenji Taniguchi, Hiroshi Yamashita, Shohei Kato

Abstract. — Let G be a reductive Lie group of Hermitian type. We investigate irreducible (unitary) highest weight representations of G which are not necessarily in the holomorphic discrete series. The results of three articles of this volume include the determination of the associated cycles, the Bernstein degrees, and the generalized Whittaker models for such representations. We give a convenient description of K-types by branching rules of representations of classical groups. An integral formula of the degrees of small nilpotent orbits is established for arbitrary Hermitian Lie algebras. The generalized Whittaker models for each unitary highest weight module are specified by means of the principal symbol of a gradient type differential operator, and also in relation to the multiplicity in the associated cycle. In the text, we also present some expository introductions of the key notions treated in this volume, such as associated cycles, Howe correspondence for dual pairs where one member of the pair is compact, and the realization of highest weight representations on the kernels of the differential operators of gradient type.

***Résumé* (Orbites nilpotentes, cycles associés et modèles de Whittaker pour les représentations de plus haut poids)**

Soit G un groupe de Lie réductif de type hermitien. Nous étudions les représentations irréductibles (unitaires) de G de plus haut poids, qui ne sont pas nécessairement dans la série discrète holomorphe. Les résultats obtenus dans les trois articles de ce volume comprennent la détermination des cycles associés, des degrés de Bernstein et des modèles de Whittaker généralisés de ces représentations. Nous donnons une description commode des K-types par les règles de branchement des représentations des groupes classiques. Une formule intégrale pour les degrés des petites orbites nilpotentes est établie pour les algèbres de Lie hermitiennes quelconques. Les modèles de Whittaker généralisés pour chaque module unitaire de plus haut poids sont spécifiés au moyen du symbole principal d'un opérateur différentiel de type gradient, et également en relation avec la multiplicité dans le cycle associé. Le texte comporte aussi des exposés introductifs concernant les principales notions considérées : cycles associés, correspondance de Howe dans le cas où la paire duale contient un membre compact et réalisation des représentations de plus haut poids dans les noyaux d'opérateurs différentiels de type gradient.

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INTRODUCTION TO THIS VOLUME

1. Associated cycle

Let G be a reductive group over $\mathbb{R}$, and consider an irreducible admissible representation π of G. There are many kinds of invariants attached to π in order to study, even classify, such representations. Among these invariants, one of the most important ones is the global character Θ_π of π. However, since the global character determines π completely, it is hard to compute Θ_π explicitly. Besides it, there are several invariants which are easier to handle; such as the infinitesimal character λ_π, the Gelfand-Kirillov dimension $\operatorname{Dim}\pi$, the Bernstein degree $\operatorname{Deg}\pi$, minimal K-types, etc., where K is a maximal compact subgroup of G. These invariants are "coarse" in the sense that a single invariant cannot specify π by itself. However, they are strong enough when you use them together to analyze the properties of π.

It is Vogan who was first aware of the importance of using the *associated variety* $\mathcal{AV}_\pi$ of π to study admissible representations of a real reductive group G ([**18, 19**]). Let $\mathfrak{g}_\mathbb{R}$ be the Lie algebra of G, and let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ of $\mathfrak{g}_\mathbb{R}$. The associated variety $\mathcal{AV}_\pi$ is defined to be the support of graded $S(\mathfrak{g})$-module $\operatorname{gr} X_\pi$ corresponding to the Harish-Chandra $(U(\mathfrak{g}), K)$-module X_π of π, where $\operatorname{gr} X_\pi$ is defined through a good filtration of X_π compatible with the natural filtration of $U(\mathfrak{g})$, and $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$ denotes the symmetric algebra of $\mathfrak{g}$ (see [**19**] for precise definition). The associated variety is a kind of geometric counterpart of the purely algebraic notion of primitive ideals. It is not so hard to compute, but, as an invariant of π, it contains rich information on π. Later, Vogan refined the notion of associated variety to define the *associated cycle*. Let us see what is the associated cycle of π briefly (for precise definition, see [**19**], and also [**NOT**] in this volume). Before that, we need some notation.

Fix a maximal compact subgroup K of G. The choice of K determines a complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We denote by $K_\mathbb{C}$ the complexification of K, which has the Lie algebra $\mathfrak{k}$. Let $\mathcal{N}(\mathfrak{p})$ be the nilpotent variety in $\mathfrak{p}$. By the adjoint representation, the group $K_\mathbb{C}$ acts on $\mathcal{N}(\mathfrak{p})$ with finitely many orbits.

The associated variety of an irreducible admissible representation π is a union of the closure of equi-dimensional nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$:

$$\mathcal{AV}_\pi = \bigcup_{i=1}^{l} \overline{\mathcal{O}_i},$$

where $\{\mathcal{O}_i\}_{i=1}^{l} \subset \mathcal{N}(\mathfrak{p})/K_{\mathbb{C}}$ is a family of $K_{\mathbb{C}}$-orbits which generate the same nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_\pi^{\mathbb{C}}$ in $\mathfrak{g}$ ($G_{\mathbb{C}}$ is a connected Lie group with Lie algebra $\mathfrak{g}$). Then, the associated cycle of π is a linear combination of the closure of $\mathcal{O}_i$:

$$\mathcal{AC}_\pi = \sum_{i=1}^{l} m_i[\overline{\mathcal{O}_i}],$$

where m_i is a positive integer called the *multiplicity* of π at $\mathcal{O}_i$. Roughly speaking, the orbits $\mathcal{O}_i$ describe the "directions" in which π spreads most rapidly (cf. [**5**]). The multiplicity m_i gives the "rank" of $U(\mathfrak{g})$-module X_π localized at $\overline{\mathcal{O}_i}$. Take $\lambda \in \mathcal{O}_i$ and let $K_{\mathbb{C}}(\lambda)$ be the fixed subgroup of $K_{\mathbb{C}}$ at λ. Then $K_{\mathbb{C}}(\lambda)$ acts on the space of multiplicities, and therefore m_i can be interpreted as the dimension of the representation of $K_{\mathbb{C}}(\lambda)$ (see [**19**, Definition 2.12]).

The cycle $\mathcal{AC}_\pi$ behaves very well as an invariant of π. For example, the orbits $\mathcal{O}_i$ are equi-dimensional, and their complex dimension is equal to the Gelfand-Kirillov dimension $\operatorname{Dim}\pi$. Also the Bernstein degree is expressed as

$$\operatorname{Deg}\pi = \sum_{i=1}^{l} m_i \deg \overline{\mathcal{O}_i},$$

where $\deg \overline{\mathcal{O}_i}$ denotes the degree of the nilpotent cone $\overline{\mathcal{O}_i}$, and it should be understood as that of the corresponding projectivised variety in $\mathbb{P}(\mathfrak{p})$ (cf. [**NOT**]; see [**4, 6**] for the definition of the degree of a projective variety).

The authors of the first article in this volume, namely, Nishiyama, Ochiai and Taniguchi (abbreviated as NOT in the following), were interested in the computation of the Bernstein degree $\operatorname{Deg}\pi$. It seemed rather hard to compute $\operatorname{Deg}\pi$ for a particular instance of π. It is directly related to the associated cycle, but only few (non-trivial) examples were known at that time.

Assume that G/K is an irreducible Hermitian symmetric space, and take an irreducible unitary *highest weight* representation π of G. As a representation of $K_{\mathbb{C}}$, the space $\mathfrak{p}$ decomposes into two irreducible components:

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

Then it is well known that $\mathcal{AV}_\pi$ is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}_\pi \subset \mathfrak{p}^+$. Hence the associated cycle can be written as

$$\mathcal{AC}_\pi = m_\pi[\overline{\mathcal{O}_\pi}] \quad (m_\pi \in \mathbb{Z}_{>0}),$$

where m_π is the multiplicity of π at $\mathcal{O}_\pi$.

In [**NOT**], NOT derive an explicit K-type formula of π via the method of reductive dual pairs in the stable range, where G is assumed to be classical. The asymptotics of the K-types implies a formula for $\mathrm{Deg}\,\pi = m_\pi \deg \overline{\mathcal{O}_\pi}$. Further, the multiplicity m_π is interpreted as the dimension of an irreducible finite dimensional representation σ of a compact Lie group G' which forms a reductive dual pair (G, G') with G. The representation σ naturally determines π through the Weil (or oscillator) representation, and vice versa. The correspondence between σ and π is called the *theta correspondence* (see below for the precise formulation which requires metaplectic covers). As a byproduct, NOT also get an integral formula of $\deg \overline{\mathcal{O}_\pi}$, which can be calculated explicitly.

For any real reductive group G, the author of the second article, Yamashita, has been interested in the embeddings of irreducible admissible representations π into a series of representations induced from certain nilpotent subgroups of G. Such an induced module is called a *generalized Gelfand-Graev representation* of G (cf. [**9**]). By construction, it is attached to each nilpotent G-orbit $\mathcal{O}^{\mathbb{R}}$ in the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ through the Dynkin-Kostant theory. We say that π has a *generalized Whittaker model* of type $\mathcal{O}^{\mathbb{R}}$ if there exists an embedding of π into the generalized Gelfand-Graev representation attached to $\mathcal{O}^{\mathbb{R}}$.

The existence of generalized Whittaker models (or such vectors) reflects some regularity of the irreducible representation π of G in question. For example, as is shown by Kostant (for quasi-split groups) and Matumoto (for any real reductive groups), π has the largest possible Gelfand-Kirillov dimension if and only if the algebraic dual of the Harish-Chandra module of π has nonzero Whittaker vectors attached to the principal nilpotent orbits (see [**NOT**, Th.2.4]). Further, Matumoto ([**11**], [**12**]) established some results of this nature on generalized Whittaker vectors in connection with the associated variety $\mathcal{AV}(\mathrm{Ann}_{U(\mathfrak{g})} X_\pi)$ of the primitive ideal $\mathrm{Ann}_{U(\mathfrak{g})} X_\pi$, or the wave front set $\mathrm{WF}(\pi)$ of π. For details, we refer to [**Y**, Introduction]. It is well-known that $\mathcal{AV}(\mathrm{Ann}_{U(\mathfrak{g})} X_\pi)$ is the closure of a single nilpotent orbit in $\mathfrak{g}$ which contains $\mathcal{AV}_\pi$. The wave front set $\mathrm{WF}(\pi)$ describes the singularity of the distribution character Θ_π of π, and it is a union of some nilpotent G-orbits in $\mathfrak{g}_{\mathbb{R}}$. By Rossmann [**14**], it is shown that $\mathrm{WF}(\pi)$ coincides with the asymptotic support of π introduced in [**1**]. Recently, Schmid and Vilonen proved that the wave front "cycle", which is a refinement of $\mathrm{WF}(\pi)$, corresponds to the associated cycle via Kostant-Sekiguchi correspondence ([**16**]).

Then, it is natural to ask whether the associated cycle characterizes the generalized Whittaker models of interest. At first glance, this problem may seem to be more difficult to handle directly, since the associated cycle lives in $\mathcal{N}(\mathfrak{p})$, contrary to the above two invariants $\mathcal{AV}(\mathrm{Ann}_{U(\mathfrak{g})} X_\pi)$ and $\mathrm{WF}(\pi)$ of π. But, in [**5**], Gyoja and Yamashita found evidence for a strong relationship between the associated variety $\mathcal{AV}_\pi$ and the embeddings in question. Moreover, for any unitary highest weight representation π