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Astérisque, tome 274 (2001)

<http://www.numdam.org/item?id=AST_2001__274__R1_0>

ASTÉRISQUE 274

CRACKTIP IS A GLOBAL MUMFORD-SHAH MINIMIZER

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Société Mathématique de France 2001
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 ***Mathematics Subject Classification.*** — 49K99, 49Q20.

Key words and phrases. — Mumford-Shah functional.

CRACKTIP IS A GLOBAL MUMFORD-SHAH MINIMIZER

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Abstract. — We show that the pair (u, K) given by $K = (-\infty, 0] \subset \mathbb{R}^2$ and

$$u(r\cos\theta, r\sin\theta) = \sqrt{2/\pi}\ r^{1/2}\sin(\theta/2) \text{ for } r > 0 \text{ and } -\pi < \theta < \pi$$

is a global Mumford-Shah minimizer. This means that if $\widetilde{K}$ is another closed set in the plane with locally finite Hausdorff measure, $\widetilde{u}$ is a function on $\mathbb{R}^2 \setminus \widetilde{K}$ with a derivative in $L^2_{\text{loc}}(\mathbb{R}^2 \setminus \widetilde{K})$, and the pair $(\widetilde{u}, \widetilde{K})$ coincides with (u, K) out of some disk B, then

$$H^1(K \cap B) + \int_{B \setminus K} |\nabla u|^2 \leqslant H^1(\widetilde{K} \cap B) + \int_{B \setminus G} |\nabla \widetilde{u}|^2,$$

where H^1 denotes Hausdorff measure.

We shall also show that every global Mumford-Shah minimizer (u', K') that is sufficiently close to (u, K) near infinity must be equivalent to it. This is the case, for instance, if some blow-in limit of (u', K') equals (u, K).

The proofs will be based on a detailed study of the harmonic function v' conjugated to u', and its level sets. We shall also use blow-up techniques and the monotonicity of an energy integral.

Résumé **(Cracktip est un minimum de Mumford-Shah global)**

Le résultat principal de ce texte est que le couple (u, K) défini par $K =]-\infty, 0] \subset \mathbb{R}^2$ et

$$u(r\cos\theta, r\sin\theta) = \sqrt{2/\pi}\, r^{1/2} \sin(\theta/2) \text{ pour } r > 0 \text{ et } -\pi < \theta < \pi$$

est un minimum global de la fonctionnelle de Mumford-Shah. Ceci signifie que si $\widetilde{K}$ est un fermé du plan de mesure de Hausdorff de dimension 1 localement finie, $\widetilde{u}$ est une fonction définie sur $\mathbb{R}^2 \setminus \widetilde{K}$ dont la dérivée est dans $L^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \widetilde{K})$, et si le couple $(\widetilde{u}, \widetilde{K})$ coïncide avec (u, K) hors d'un disque B, on a

$$H^1(K \cap B) + \int_{B \setminus K} |\nabla u|^2 \leqslant H^1(\widetilde{K} \cap B) + \int_{B \setminus G} |\nabla \widetilde{u}|^2,$$

où l'on note H^1 la mesure de Hausdorff.

On montrera aussi que tout minimum global (u', K') de la fonctionnelle de Mumford-Shah qui est suffisament proche de (u, K) à l'infini lui est équivalent. C'est le cas par exemple si l'une des limites de (u', K') par implosions est égale à (u, K).

La démonstration est basée sur une étude détaillée de la fonction harmonique conjuguée de u' et de ses ensembles de niveau. On utilise aussi des techniques d'explosion et la monotonie d'une intégrale d'énergie.

CONTENTS

CHAPTER A

GENERAL INTRODUCTION

1. Introduction

The main goal of this paper is to verify that cracktips (as defined below) are global minimizers of the Mumford-Shah functional. This gives a positive answer to a question of E. De Giorgi [DG]. We shall also prove that all the global minimizers of the Mumford-Shah functional that are close enough to cracktips (in ways that will soon be made precise) are cracktips.

The global version of the Mumford-Shah functional that we consider here is morally given by

$$J(u,K) = \int_{\mathbb{R}^2 \setminus K} |\nabla u|^2 + H^1(K), \tag{1.1}$$

where $H^1(K)$ denotes the one-dimensional Hausdorff measure of the closed set K [see for instance [Fa] or [Ma] for definitions]. This is only a moral definition, because $J(u,K) = +\infty$ for all interesting competitors (u,K), and so we shall have to give a more local definition of competitors and minimizers.

Let us first define the set U_0 of *admissible pairs* (or competitors). These are the pairs (u,K), where K is a closed subset of the plane, $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \setminus K)$ is a function defined on $\mathbb{R}^2 \setminus K$ and whose distributional gradient ∇u lies in $L^2_{\text{loc}}\big(\mathbb{R}^2 \setminus K\big)$, and which satisfy the additional requirement that

$$H^1(K \cap B_R) + \int_{B_R \setminus K} |\nabla u|^2 < +\infty \text{ for all } R > 0, \tag{1.2}$$

where we denote by $B_R = B(0,R)$ the open disk centered at the origin and with radius R.

Next let $(u,K) \in U_0$ be given. A *competitor for* (u,K) is an admissible pair $(v,G) \in U_0$ which satisfies the following properties for some (large) $R > 0$:

$$G \setminus B_R = K \setminus B_R, \tag{1.3}$$

$$(1.4) \qquad v(x) = u(x) \quad \text{on} \quad \mathbb{R}^2 \setminus (K \cup B_R),$$

and

$$(1.5) \qquad \text{if } x, y \in \mathbb{R}^2 \setminus (K \cup B_R) \text{ are separated by } K, \text{then } G \text{ also separates them.}$$

[We say that K separates x from y when x, y lie in different connected components of $\mathbb{R}^2 \setminus K$.]

Note that if (1.3)-(1.5) hold for $R > 0$, they also hold for all $R' > R$.

Definition 1.6. — A *global minimizer* (for the Mumford-Shah functional) is an admissible pair $(u, K) \in U_0$ such that

$$(1.7) \qquad H^1(K \cap B_R) + \int_{B_R \setminus K} |\nabla u|^2 \leqslant H^1(G \cap B_R) + \int_{B_R \setminus G} |\nabla v|^2$$

for all competitors (v, G) for (u, K) and $R > 0$ such that (1.3)-(1.5) hold.

Note that we do not need to be too specific about R: if (1.7) holds for some R such that (1.3)-(1.5) hold, it stays true for all such R (by (1.3) and (1.4)).

This class of global minimizers was introduced in [Bo], where it is shown that if (u_0, K_0) is a minimizer for the (usual) Mumford-Shah functional

$$(1.8) \qquad J(u, K) = \int_{\Omega \setminus K} |u - g|^2 + H^1(K) + \int_{\Omega \setminus K} |\nabla u|^2$$

(on a bounded domain Ω, and with a given bounded function g on Ω), then all limits of (u, K) under blow-ups are global minimizers as in Definition 1.6. Our topological condition (1.5) actually comes from this in a natural way; see Section 12 for details about this.

Note that if (u, K) is a global minimizer, we can always add any closed set of H^1-measure zero to K, and this gives another global minimizer equivalent to (u, K). We shall say that the global minimizer (u, K) is "*reduced*" if there is no proper closed subset $\widetilde{K}$ of K such that u extends to a function $\widetilde{u} \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \setminus \widetilde{K})$ and $(\widetilde{u}, \widetilde{K})$ is a competitor for (u, K). [We add this last constraint because of the topological condition (1.5); we want to avoid opening holes that would change the true nature of (u, K).]

It is not too hard to check that for each global minimizer (u, K) there is a reduced global minimizer $(\widetilde{u}, \widetilde{K})$ which is equivalent to (u, K), that is, such that $\widetilde{K} \subset K$, $\widetilde{u}$ is an extension of u, and $(\widetilde{u}, \widetilde{K})$ is a competitor for (u, K). Because of this, we shall always assume that all our global minimizers are reduced. We don't loose any generality, and this will allow us to give more pleasant descriptions of K.

The natural analogue in the present context of the celebrated Mumford-Shah conjecture in [MuSh] is that all (reduced) global minimizers belong to the following short list:

$$(1.9) \qquad K = \varnothing \quad \text{and } u \text{ is constant;}$$

$$\begin{cases} K \text{ is a line and } u \text{ is constant on each} \\ \text{of the two connected components of } \mathbb{R}^2 \setminus K \ ; \end{cases} \tag{1.10}$$

$$\begin{cases} K \text{ is a propeller (see the definition below) and} \\ u \text{ is constant on each of the 3 components of } \mathbb{R}^2 \setminus K \ ; \end{cases} \tag{1.11}$$

$$(u, K) \text{ is a cracktip (also see below).} \tag{1.12}$$

We call propeller a union of three half-lines with a common endpoint (called center) and that make 120° angles with each other.

We call cracktips the pairs (u, K) such that, after a suitable change of coordinates in the plane,

$$K = \{(x, 0) \ ; \ x \leqslant 0\} \tag{1.13}$$

(a half-line) and u is given by

$$u\,(r\cos\theta, r\sin\theta) = \pm\sqrt{2/\pi}\ r^{1/2}\sin\frac{\theta}{2} + C \tag{1.14}$$

for $r > 0$ and $-\pi < \theta < \pi$. The (constant) sign $\pm$ and the value of the constant C obviously do not matter.

If the conjecture above where true, then the Mumford-Shah conjecture would quite probably follow, using the arguments in [Bo] (we did not check all the details). The converse is less clear a priori (there could be global minimizers that do not show up as blow-ups of Mumford-Shah minimizers).

Recall that if (u, K) is a global minimizer and K is connected, then (u, K) belongs to the short list above. This is one of the main points of [Bo], where it is used to prove that isolated components of K_0 for Mumford-Shah minimizers are finite unions of C^1-curves.

The pairs (u, K) in (1.9), (1.10), and (1.11) are easily seen to be global minimizers. Note that the topological condition (1.5) is needed for this. The main result of this paper is that

(1.15) Cracktips (as defined by (1.13) and (1.14)) are global minimizers.

Note that in the case of cracktips, the topological condition (1.5) on competitors for (u, K) is void, because K does not separate any pair of points. Thus cracktips are also minimizers for De Giorgi's definition, even though [DG] does not mention (1.5).

Our proof of (1.15) will also give that all global minimizers that are sufficiently close to a cracktip are actually cracktips. The precise meaning of "sufficiently close" may vary a little. Here is an example of sufficient condition.