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**FAMILIES TORSION
AND MORSE FUNCTIONS**

**Jean-Michel Bismut
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FAMILIES TORSION AND MORSE FUNCTIONS

Jean-Michel Bismut, Sebastian Goette

Abstract. — To a flat vector bundle, one can associate odd real characteristic classes. Bismut and Lott have proved a Riemann-Roch-Grothendieck theorem for such classes, when taking the direct image of a flat vector bundle by a proper submersion. They have also constructed associated secondary invariants, the analytic torsion forms in de Rham theory. The component of degree 0 of these forms is the classical Ray-Singer torsion.

The present paper has five purposes:

- to extend the theory of analytic torsion forms to the equivariant setting.
- to give a proper normalization of these torsion forms.
- to prove rigidity formulas, showing that in positive degree, and up to locally computable terms, these forms are rigid under deformation of the flat connection.
- to evaluate the equivariant analytic torsion forms modulo coboundaries, under the assumption that there exists a fibrewise gradient vector field which verifies the Morse-Smale transversality conditions in every fibre.
- to compute the equivariant analytic torsion forms of sphere bundles associated to vector bundles.

Our main formula generalizes the results by Cheeger, Müller, Lott-Rothenberg and Bismut-Zhang on the relation of Ray-Singer torsion to Reidemeister torsion, and also computations by Bunke for sphere bundles.

Résumé (Torsion en famille et fonctions de Morse). — À un fibré plat, on peut associer des classes caractéristiques impaires réelles. Bismut et Lott ont montré un théorème de Riemann-Roch-Grothendieck, quand on prend l'image directe d'un fibré plat par une submersion propre. Ils ont aussi construit des invariants secondaires, les formes de torsion analytique en théorie de de Rham, qui sont des formes paires sur la base de la fibration considérée. La composante de degré 0 de ces formes est la torsion analytique de Ray-Singer.

Le présent article a pour objet :

- d'étendre la théorie des formes de torsion analytique en situation équivariante.
- de normaliser les formes de torsion analytique.
- d'établir des résultats de rigidité, qui montrent qu'à des termes explicites calculables localement près, les formes de torsion ne varient pas par déformation de la connexion plate considérée, et ceci en degré positif.
- d'évaluer les formes de torsion analytique équivariantes, sous l'hypothèse qu'il existe un champ de gradient de Morse-Smale dans les fibres.
- d'évaluer les formes de torsion équivariantes des fibrés en sphères provenant de fibrés vectoriels.

Le résultat principal généralise des résultats obtenus par Cheeger, Müller, et Lott-Rothenberg et Bismut-Zhang sur le lien entre torsion analytique et torsion de Reidemeister, et aussi des calculs de Bunke pour des fibrés en sphères.

CONTENTS

Introduction	1
1. Flat superconnections and equivariant torsion forms	13
1.1. The superconnection formalism	13
1.2. The transpose of a superconnection	14
1.3. The adjoint of a superconnection	15
1.4. Superconnections and group actions	15
1.5. Flat superconnections, Hermitian metrics and odd closed forms	16
1.6. Superconnections of total degree 1	18
1.7. A rescaled metric	20
1.8. The limit of $h_g(A', g_t^E)$ and of $h^\wedge(A', g_t^E)$ as $t \rightarrow +\infty$	21
1.9. The form $S_{h,g}(A', g^E)$	22
1.10. Flat complexes of vector bundles and their torsion forms	23
1.11. Functorial characterization of the torsion forms	25
2. Rigidity of torsion forms and their Chern normalization	27
2.1. Rigidity properties of the superconnection odd classes	27
2.2. An expression for $k(D_t)$	30
2.3. A convergence result	33
2.4. Rigidity properties of the forms $S_{h,g}(A', g^E)$	39
2.5. Rigidity properties of the torsion forms $T_{h,g}(A', g^E)$	39
2.6. The imaginary part of the odd Chern classes	41
2.7. Superconnection classes and the Chern character	44
2.8. The Chern torsion forms	48
2.9. Generalized metrics and the forms $U_{h,g}(A', g^E)$	48
2.10. Generalized metrics and flat complexes	51

3. Analytic torsion forms : rigidity and the Chern character	55
3.1. Equivariant smooth fibrations	56
3.2. A flat superconnection of total degree 1	57
3.3. A metric on TX and the tensors T and S	58
3.4. The adjoint superconnection	59
3.5. Clifford algebras	61
3.6. The Levi-Civita superconnection	61
3.7. A rescaling of the metric on TX	63
3.8. A Lichnerowicz formula	64
3.9. A unitary connection on $H^\bullet(X, F _X)$	66
3.10. The odd closed forms $h_g\left(A', g_t^{\Omega^\bullet(X, F _X)}\right)$	67
3.11. A transgression formula	69
3.12. The equivariant analytic torsion forms	70
3.13. Analytic torsion forms associated to arbitrary functions	72
3.14. An identity for $k(D_t)$	73
3.15. A convergence result	76
3.16. Rigidity formulas for the analytic torsion forms	77
3.17. The Chern analytic torsion forms	78
4. The analytic torsion forms of a $\mathbf{Z}_2$-graded vector bundle	79
4.1. The flat superconnection of a $\mathbf{Z}_2$ -graded vector bundle	80
4.2. The superconnection heat kernel and the function σ	82
4.3. The asymptotics of the heat equation supertraces	87
4.4. The higher analytic torsion forms of a $\mathbf{Z}_2$ -graded vector bundle	88
4.5. The Chern analytic torsion forms of a $\mathbf{Z}_2$ -graded vector bundle	91
4.6. The Lerch series and the function $I(\theta, x)$	91
4.7. The Lerch series and the function $J(\theta, x)$	97
4.8. Formal relation to the R genus	100
5. A family of Thom-Smale gradient vector fields	101
5.1. The Thom-Smale complex of a gradient vector field	101
5.2. The de Rham map of Laudenbach	105
5.3. Equivariant Thom-Smale complexes	106
5.4. A smooth fibration	108
5.5. A family of gradient vector fields	108
5.6. A families version of the results of Laudenbach	110
6. Fibrations, Berezin integrals and Euler currents	117
6.1. The Berezin integral	117
6.2. The Mathai-Quillen Thom forms	118
6.3. Convergence of the Mathai-Quillen currents	120
6.4. A transgressed Euler class	120
6.5. Fibrations and curvature identities	122
6.6. A Stokes formula	125

7. Analytic torsion forms and Morse-Smale vector fields	127
7.1. Assumptions and notation	127
7.2. Statement of the main result	129
7.3. The formula for the Chern analytic torsion forms	129
7.4. Compatibility of Theorem 7.2 to deformations and rigidity results on analytic torsion forms	130
7.5. Compatibility of Theorem 7.2 to products	131
7.6. Compatibility of Theorem 7.2 to Poincaré duality	133
7.7. Changing the Morse gradient field	134
7.8. The case where $g^{F B}$ is flat and $\dim X_g$ is odd	135
7.9. The case where $g^{F B}$ is flat and $\dim X_g$ is even	137
7.10. The case where $g^{F B}$ is flat, $\dim X$ is odd and F is acyclic	138
8. A contour integral	141
8.1. A closed form	141
8.2. A contour integral	142
9. A proof of the main result	145
9.1. Some simplifying assumptions on the metrics g^{TX}, g^F	145
9.2. Convergence results on integrals of differential forms	146
9.3. Seven intermediate results	149
9.4. The asymptotics of the I_k^0	151
9.5. Matching the divergences	157
9.6. A proof of Theorem 7.2	158
10. Generalized metrics : a first proof of Theorem 9.8	159
10.1. The harmonic oscillator near B	160
10.2. The eigenbundles associated to small eigenvalues	162
10.3. The grading of A_T^2	166
10.4. The projectors $P_T^{[0,1]}$	167
10.5. The projectors $P_{t,T}^{[0,1]}$	168
10.6. The maps P_T^∞	171
10.7. The generalized metric $g_T^{C^\bullet(W^u, F)}$	173
10.8. The maps $P_{t,T}^\infty$ and the generalized metrics $g_{t,T}^{C^\bullet(W^u, F)}$	174
10.9. Replacing M by $M \times \mathbf{R}_+^*$	179
10.10. The superconnection forms for $F_T^{[0,1]}$	181
10.11. The form c and the complex $C^\bullet(W^u, F)$	183
10.12. An identity on the forms $U_{h,g} \left(A^{C^\bullet(W^u, F)'} , g_{t,T}^{C^\bullet(W^u, F)} \right)$	185
10.13. A fundamental result	187
10.14. The projectors $\bar{P}_T$	187
10.15. The maps J_T and $\bar{e}_T$	189
10.16. A proof of the first part of Theorem 10.50	194
10.17. A proof of the second part of Theorem 10.50	196

11. Fibrewise nice functions : a second proof of Theorem 9.8	197
11.1. An extra simplifying assumption	197
11.2. Two fundamental results	198
11.3. Preliminary results	198
11.4. The spectrum of $B_T^{(0)}$	200
11.5. The superconnection supertraces associated to $C^\bullet(W^u, F)$	201
11.6. The superconnection supertraces associated to $F_T^{[0,1]}$	204
11.7. Two intermediate results	205
11.8. The term containing $F_{t,T}$	208
11.9. The term containing $G_{t,T}$	210
11.10. The term containing $H_{t,T}$	214
11.11. A compatibility result	215
11.12. A Proof of Theorem 11.1	216
11.13. A proof of Theorem 11.2	217
12. An asymptotic expansion for $\text{Tr}_s [fgh'(D_{t,T})]$ as $T \rightarrow +\infty$	219
12.1. A Lichnerowicz formula	219
12.2. A proof of Theorem 9.9	220
12.3. An estimate on the kernel of $\exp\left(-\overline{C}_{t,T}^2\right)$	220
12.4. A proof of Theorem 12.2	224
12.5. A proof of Theorem 9.7	226
13. The asymptotics of $\text{Tr}_s \left[fgh' \left(D_{t,T/\sqrt{t}} \right) \right]$ as $t \rightarrow 0$	229
13.1. A convergence result and a proof of Theorem 9.10	230
13.2. Localization of the problem	231
13.3. Replacing X by TX	234
13.4. The Getzler rescaling	236
13.5. The first term in the asymptotics of $\text{Tr}_s \left[fg \exp \left(-C_{t,T/\sqrt{t}}^2 \right) \right]$	238
13.6. The asymptotic expansion of the operator $L_{t,T}^{3,x}$	239
13.7. A technical result	241
13.8. A proof of Theorem 13.1 for bounded T	242
13.9. A proof of Theorem 13.19	245
13.10. A proof of Theorem 13.1	251
14. The asymptotics of $\text{Tr}_s [fgh'(D_{t,T/t})]$ as $t \rightarrow 0$	259
14.1. A convergence result and a proof of Theorem 9.11	259
14.2. Localization of the problem near X_g	260
14.3. An estimate away from B_g	261
14.4. A proof of Theorem 14.1	262
15. The asymptotics of $\text{Tr}_s [fgh'(D_{t,T/t})]$ as $T \rightarrow +\infty$	265
15.1. A convergence result and a proof of Theorem 9.12	265
15.2. An estimate on the kernel of $\exp\left(-\overline{C}_{t,T/t}^2\right)$	265
15.3. A proof of Theorem 15.1	267

16. The analytic torsion forms of unit sphere bundles	269
16.1. A formula for the analytic torsion forms of a unit sphere bundle	270
16.2. The suspension of a sphere and a Morse-Bott function	270
16.3. The torsion forms of $S^{E \oplus \mathbf{R}}$	274
16.4. The embedding formula	275
16.5. A proof of Theorem 16.11	275
16.6. Adiabatic limits : the case where n is odd	282
Bibliography	285
Index	291