Astérisque

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Astérisque, tome 288 (2003)

<http://www.numdam.org/item?id=AST_2003__288__R1_0>

ASTÉRISQUE 288

VIEWS OF PARAMETER SPACE: TOPOGRAPHER AND RESIDENT

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Société Mathématique de France 2003
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification.* — 37J40, 37J50, 32H50, 57NXX, 57N16, 30F60.

Key words and phrases. — Rational maps, branched coverings, critical point, Teichmüller space.

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Abstract. — In this work, we investigate the structure of certain parameter spaces. The aim is to understand the variation of dynamics — in particular, of hyperbolic dynamics — in certain parameter spaces of rational maps. In order to do this, we examine the topological and geometric structure of larger parameter spaces, of branched coverings of the Riemann sphere $\overline{\mathbf{C}}$, where some of the critical points are constrained to have finite forward orbits.
We obtain a complete topological description of the spaces under consideration, from two points of view, which we call the *Topographer's View* and the *Resident's View*. The *Topographer's View* is, in essence, a geometrising theorem. It shows that the space in question is, up to homotopy equivalence, a countable union of disjoint geometric pieces, joined together by handles. The most typical geometric pieces are varieties of rational maps, and tori. The *Resident's View* is a view of the whole parameter space from the dynamical plane of a map (a resident) in the parameter space. This is necessarily a two-dimensional view, in which the geometric pieces of the parameter space appear as disjoint convex regions in the dynamical plane.

***Résumé* (Points de vue sur l'espace de paramètres: le topographe et le résident)**

Dans ce travail, nous étudions la structure de certains espaces de paramètres. L'objectif est de comprendre les variations de dynamique — en particulier de dynamique hyperbolique — dans certains espaces paramétrant des applications rationnelles. Pour cela, nous examinons la structure topologique et géométrique d'espaces plus grands paramétrant des revêtements ramifiés de la sphère de Riemann $\overline{\mathbf{C}}$, où plusieurs points critiques sont contraints à avoir une orbite positive finie.

Nous obtenons une description topologique complète des espaces considérés, de deux points de vue, que nous appelons la *vue du topographe* et la *vue du résident*. La vue topographique est, en somme, un théorème de géométrisation. Elle montre que l'espace en question est, à une équivalence d'homotopie près, une réunion dénombrable de morceaux géométriques disjoints, reliés ensembles par des anses. Les morceaux géométriques les plus typiques sont des variétés d'applications rationnelles et des tores. La *vue du résident* est une vue de l'espace des paramètres tout entier depuis le plan dynamique d'une application (un résident) situé dans l'espace des paramètres. C'est nécessairement une vue en dimension 2, dans laquelle les morceaux géométriques de l'espace des paramètres apparaissent comme des régions convexes disjointes dans le plan dynamique.

CONTENTS

PART I

TOPOLOGY, COMBINATORICS, VIEWS

INTRODUCTION

The objects of study in this paper are rational maps of the Riemann sphere $\overline{\mathbf{C}}$, considered as dynamical systems. The basic problem is to understand variation of dynamics in a given family of rational maps. The total dynamics of a rational map is greatly influenced by the dynamical behaviour of its critical points. So it is natural to consider families of rational maps in which some critical points are constrained to be periodic, or eventually periodic. Thus, we wish to study a parameter space of dynamical systems, with specified dynamics on some invariant set which varies isotopically throughout the parameter space. A rational map with a finite invariant set is a holomorphic map of a marked Riemann surface. So our object of study is a topological space in which the points are both dynamical systems and geometric structures.

Paths are important in topology. When the points in a topological space M are themselves mathematical objects, then paths in the space reflect this additional structure. For example, let S_0 be a compact topological surface, $M = M(S_0)$ the moduli space of Riemann surfaces homeomorphic to S_0, and let $S \in M$. We get different views of S from the endpoints of a closed homotopically nontrivial path in M based at S. If we wish to understand a space of mathematical objects, then we need an understanding of the different views of each mathematical object. This involves understanding the extra structure inherited by paths in M when the points in M have additional structure. For example, if M is as above, then closed paths in M based at S, which avoid singular points, give rise to homeomorphisms of S, modulo isotopy.

Study of any parameter space of dynamical systems involves looking at relative movement of points in the dynamical plane as a point moves in parameter space. This simple-minded idea manifests itself in virtually every paper written on dynamical systems. Sometimes the study of relative movement is local, as in, for example, basic theory of persistence (or otherwise) of fixed points and corresponding local dynamics. Sometimes it is global, as, for example in study of the Mandelbrot set for quadratic