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Peter Scott, Gadde A. Swarup

Abstract. — We find canonical decompositions for (almost) finitely presented groups which essentially specialise to the classical JSJ-decomposition when restricted to the fundamental groups of Haken manifolds. The decompositions that we obtain are invariant under automorphisms of the group. A crucial new ingredient is the concept of a regular neighbourhood of a family of almost invariant subsets of a group. An almost invariant set is an analogue of an immersion.

***Résumé* (Voisinages réguliers et décompositions canoniques pour les groupes)**
Nous définissons une décomposition canonique pour les groupes presque finiment présentés qui correspond à la décomposition JSJ classique dans le cas du groupe fondamental d'une variété de Haken. Les automorphismes du groupe laissent invariante cette décomposition. Un élément crucial et nouveau est le concept de voisinage régulier d'une famille de sous-ensembles du groupe qui sont presque invariants. Un ensemble presque invariant est un analogue d'une immersion.

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Appendix

INTRODUCTION

This article is devoted to the study of analogues for groups of the classical JSJ-decomposition (see Jaco and Shalen [**25**], Johannson [**26**] and Waldhausen [**56**]) for orientable Haken 3–manifolds. The orientability restriction is not essential but it will simplify our discussions. An announcement of our results is in [**46**]. This field was initiated by Kropholler [**27**] who studied analogous decompositions for Poincaré duality groups of any dimension greater than 2. But the current interest in this kind of decomposition started with the work of Sela [**49**] on one-ended torsion-free hyperbolic groups. His results were generalised by Rips and Sela [**36**], Bowditch [**5**][**8**], Dunwoody and Sageev [**14**], Dunwoody and Swenson [**15**], and Fujiwara and Papasoglu [**20**], but none of these results yields the classical JSJ-decomposition when restricted to the fundamental group of an orientable Haken manifold. In this paper, we give a new approach to this subject, and we give decompositions for finitely presented groups which essentially specialise to the classical JSJ-decomposition when restricted to the fundamental groups of Haken manifolds. An important feature of our approach is that the decompositions we obtain are unique and are invariant under automorphisms of the group. In previous work such strong uniqueness results were only found for decompositions of word hyperbolic groups. Most of the results of these previous authors for virtually polycyclic groups can be deduced from our work. But our arguments use some of the results of these authors, particularly those of Bowditch. In addition, we use the important work of Dunwoody and Roller in [**13**]. Our work also yields some extensions of the results on the Algebraic Annulus and Torus Theorems in [**43**], [**5**] and [**15**]. It should be remarked that even though we obtain canonical decompositions for all finitely presented groups, these decompositions are often trivial. This is analogous to the fact that any finitely generated group possesses a free product decomposition, but this decomposition is trivial whenever the given group is freely indecomposable. We should also remark that many of the ideas in this paper and the above mentioned papers can be traced back to the groundbreaking work of Stallings on groups with infinitely many ends [**52**][**53**].

We focus on what we consider to be the most important aspects of the topological JSJ-decomposition. Our choice of the crucial property of this decomposition is the Enclosing Property of the characteristic submanifold, and we use an algebraic generalisation of this property. The topological Enclosing Property can be described briefly as follows. See chapter 1 for a more detailed discussion. For an orientable Haken 3–manifold M, Jaco and Shalen [**25**] and Johannson [**26**] proved that there is a family $\mathcal{T}$ of disjoint essential annuli and tori embedded in M, unique up to isotopy, and with the following properties. The manifolds obtained by cutting M along $\mathcal{T}$ are simple or are Seifert fibre spaces or I–bundles over surfaces. The Seifert and I–bundle pieces of M are said to be *characteristic*, and any essential map of the annulus or torus into M can be properly homotoped to lie in a characteristic piece. This is called the Enclosing Property of $\mathcal{T}$. The characteristic submanifold $V(M)$ of M consists essentially (see chapter 1 for details) of the union of the characteristic pieces of the manifold obtained from M by cutting along $\mathcal{T}$. The fundamental group G of M is the fundamental group of a graph Γ of groups, whose underlying graph is dual to the frontier of $V(M)$. Thus the edge groups of Γ are all isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, and the vertex groups are the fundamental groups of simple manifolds or of Seifert fibre spaces or of surfaces. The uniqueness up to isotopy of the splitting family $\mathcal{T}$ implies that Γ is unique. Further, the Enclosing Property implies that any subgroup of G which is represented by an essential annulus or torus in M is conjugate into a characteristic vertex group.

All the previous algebraic analogues of the topological JSJ-decomposition consist of producing a graph of groups structure Γ for a given group G with the edge groups of Γ being of some specified type and with some "characteristic" vertices. The algebraic analogue of the topological Enclosing Property which was used is the property that certain "essential" subgroups of G must be conjugate into one of the characteristic vertex groups of Γ. Note that the word "essential" was not used by any of these authors, and they considered several different classes of subgroups. We use the term as a convenience to allow us to compare their differing results.

Our results also yield a graph of groups structure Γ for a given group G with some "characteristic" vertices, but our algebraic generalisation of the topological Enclosing Property corresponds more closely to the topological situation.

Here is a more detailed discussion of the previous algebraic analogues of the topological JSJ-decomposition. In all of these cases, G is a finitely presented one-ended group, and an essential subgroup of G is of the same abstract type as the edge groups of Γ. For example, when trying to describe all splittings of a group G over infinite cyclic subgroups, previous authors produced a decomposition with infinite cyclic edge groups, such that if G splits over an infinite cyclic subgroup H, then H is conjugate into a characteristic vertex group. The first such result was by Kropholler [**27**], who considered the special case when G is a Poincaré duality group of dimension n and the edge groups of Γ are virtually polycyclic (VPC) groups of Hirsch length $n-1$. For brevity, we will refer to the length rather than the Hirsch length of a VPC group throughout this paper. We will also say that a VPC group of length n is $VPCn$. In his case, any $VPC(n-1)$ subgroup H is essential. Such a subgroup H will have a subgroup K of index at most 2 such that $e(G,K) \geqslant 2$. This corresponds to considering all π_1–injective maps of closed $(n-1)$–dimensional manifolds into a n–manifold rather than considering just embeddings of such manifolds. Note that a VPC group of length at most 2 is virtually abelian, so that when $n=3$ his result is closely related to the topology of 3–manifolds. In fact, Kropholler [**28**] used his results in [**27**] to give a new proof of the existence of the JSJ-decomposition for closed 3–manifolds.

In most of the papers which came after [**27**], a subgroup H of G is essential if G possesses a splitting over H. Such subgroups correspond to embedded codimension–1 manifolds in a manifold. Sela [**48**] considered the case when G is a torsion-free word hyperbolic group, the essential subgroups are infinite cyclic and the edge groups of Γ are also infinite cyclic. Rips and Sela [**36**] generalised this to the case where G is a torsion-free finitely presented group. The essential subgroups and the edge groups are again infinite cyclic. Dunwoody and Sageev [**14**] considered the case when G is a finitely presented group and the essential subgroups and the edge groups of Γ are slender groups (i.e. groups in which every subgroup is finitely generated), subject to the constraint that if H is an edge group, then G admits no splitting over a subgroup of infinite index in H. Fujiwara and Papasoglu [**20**] considered the case when G is a finitely presented group and the essential subgroups and the edge groups of Γ are finitely generated small groups (i.e. groups which do not admit a hyperbolic action on a tree), subject to the weaker constraint that if H is an edge group, then no splitting of G over H can cross strongly a splitting of G over a subgroup of H of infinite index. (See chapter 2 for a discussion of crossing and strong crossing.) Dunwoody and Swenson [**15**] considered the case when G is a finitely presented group and the essential subgroups and the edge groups of Γ are VPC groups of a fixed length n, subject to the constraint that G admits no splitting over a VPC subgroup of length less than n. In their work, a $VPCn$ subgroup H of G is essential if $e(G,H) \geqslant 2$. This corresponds to considering singular codimension–1 manifolds in a manifold rather than just embedded ones. Finally, Bowditch [**5**] considered the case when G is a word hyperbolic group and the essential subgroups of G and the edge groups of Γ are two-ended (which is equivalent to being virtually infinite cyclic). In his work, a two-ended subgroup H of G is essential if $e(G,H) \geqslant 2$. This corresponds to considering all essential annuli in a 3–manifold rather than just embedded ones. In this case, Bowditch proved an existence and uniqueness result, precisely analogous to the 3–manifold theory in the atoroidal case.

The above results are often referred to vaguely but collectively as the JSJ-decomposition of a finitely presented group. While these results are commonly