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**BERGMAN KERNELS AND
SYMPLECTIC REDUCTION**

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BERGMAN KERNELS AND SYMPLECTIC REDUCTION

Xiaonan Ma, Weiping Zhang

Abstract. — We generalize several recent results concerning the asymptotic expansions of Bergman kernels to the framework of geometric quantization and establish an asymptotic symplectic identification property. More precisely, we study the asymptotic expansion of the G -invariant Bergman kernel of the spin c Dirac operator associated with high tensor powers of a positive line bundle on a symplectic manifold admitting a Hamiltonian action of a compact connected Lie group G . We also develop a way to compute the coefficients of the expansion, and compute the first few of them, especially, we obtain the scalar curvature of the reduction space from the G -invariant Bergman kernel on the total space. These results generalize the corresponding results in the non-equivariant setting, which have played a crucial role in the recent work of Donaldson on stability of projective manifolds, to the geometric quantization setting.

As another kind of application, we establish some Toeplitz operator type properties in semi-classical analysis in the framework of geometric quantization.

The method we use is inspired by Local Index Theory, especially by the analytic localization techniques developed by Bismut and Lebeau.

Résumé (Noyaux de Bergman et réduction symplectique). — Nous généralisons des résultats récents sur le développement asymptotique du noyau de Bergman au cadre de quantification géométrique, et établissons une propriété d'identification asymptotique symplectique. Plus précisément, nous étudions le développement asymptotique du noyau de Bergman G -invariant de l'opérateur de Dirac spin c associé à une puissance tendant vers l'infini d'un fibré en droites positif sur une variété symplectique compacte munie d'une action hamiltonienne d'un groupe de Lie compact connexe. Nous développons aussi une façon de calculer les coefficients du développement, et nous calculons les premiers termes, en particulier, nous obtenons la courbure scalaire de la réduction symplectique à partir du noyau de Bergman G -invariant sur l'espace total. Ces résultats généralisent les résultats correspondants dans le cas non-équivariant, qui ont joué un rôle crucial dans un travail récent de Donaldson sur la stabilité de variétés projectives, au cadre de quantification géométrique.

Comme application de notre développement, nous établissons aussi des propriétés de type opérateur de Toeplitz en limite semi-classique dans le cadre de quantification géométrique.

Notre méthode est inspirée par la théorie de l'indice local, en particulier les techniques de localisation analytique développées par Bismut-Lebeau.

Dedicated to our teacher Jean-Michel Bismut

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CHAPTER 0

INTRODUCTION

The study of the Bergman kernel is a classical subject in the theory of several complex variables, where usually it concerns the kernel function of the projection operator to an infinite dimensional Hilbert space. The recent interest of the analogue of this concept in complex geometry mainly started with the paper of Tian [43], which was in turn inspired by a question of Yau [46]. Here, the projection concerned is, however, onto a finite dimensional space.

Since [43], the Bergman kernel has been studied extensively in [38], [14], [47], [25], where the diagonal asymptotic expansion properties for high powers of an ample line bundle were established. Moreover, the coefficients in the asymptotic expansion encode geometric information of the underlying complex projective manifolds. This asymptotic expansion plays a crucial role in the recent work of Donaldson [18], where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to the Chow-Mumford stability.

In [17], [28], [30], Dai, Liu, Ma and Marinescu studied the full off-diagonal asymptotic expansion of the (generalized) Bergman kernel of the spin^c Dirac operator and the renormalized Bochner–Laplacian associated to a positive line bundle on a compact symplectic manifold. As a by product, they gave a new proof of the results mentioned in the previous paragraph. They found also various applications therein, especially as was pointed out in [30], the full off-diagonal asymptotic expansion implies Toeplitz operator type properties. This approach is inspired by the Local Index Theory, especially by the analytic localization techniques of Bismut-Lebeau [7, §11]. We refer to the above papers as well as the recent book [31] for detail informations of the Bergman kernel on compact symplectic manifolds.

In this paper, we generalize some of the results in [17], [28] and [30] to the framework of geometric quantization, by studying the asymptotic expansion of the G -invariant Bergman kernel for high powers of an ample line bundle on symplectic manifolds admitting a Hamiltonian group action of a compact Lie group G .