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SLOPE FILTRATIONS FOR RELATIVE FROBENIUS

by

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Abstract. — The slope filtration theorem gives a partial analogue of the eigenspace decomposition of a linear transformation, for a Frobenius-semilinear endomorphism of a finite free module over the Robba ring (the ring of germs of rigid analytic functions on an unspecified open annulus of outer radius 1) over a discretely valued field. In this paper, we give a third-generation proof of this theorem, which both introduces some new simplifications (particularly the use of faithfully flat descent, to recover the theorem from a classification theorem of Dieudonné-Manin type) and extends the result to allow an arbitrary action on coefficients (previously the action on coefficients had to itself be a lift of an absolute Frobenius). This extension is relevant to a study of (ϕ, Γ)-modules associated to families of p-adic Galois representations, as initiated by Berger and Colmez.

***Résumé* (Filtrations de pentes pour le Frobenius relatif).** — Le théorème de filtration par les pentes donne un analogue partiel de la décomposition en espaces propres d'une transformation linéaire, pour un endomorphisme semilinéaire (pour Frobenius) d'un module libre de type fini sur l'anneau de Robba (l'anneau des germes de fonctions analytiques rigides sur une couronne ouverte non précisée de rayon externe 1) sur un corps à valuation discrète. Dans cet article, nous donnons une preuve de troisième génération de ce théorème, qui introduit quelques simplifications nouvelles (en particulier, l'emploi de la descente fidèlement plate, pour obtenir le théorème à partir d'un théorème de classification de type Dieudonné-Manin). Nous étendons aussi le résultat pour permettre une action arbitraire sur les coefficients (auparavant, cette action devait être un relèvement d'un Frobenius absolu). Cette extension est utile pour l'étude des (ϕ, Γ)-modules associés à des familles de représentations galoisiennes p-adiques; Berger et Colmez ont commencé cette étude.

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Introduction

This paper describes a third-generation proof of the slope filtration theorem for Frobenius modules over the Robba ring (Theorem 1.7.1 herein). This proof is more expedient than what one finds in our original paper [**21**] or its sequel [**22**]. In addition, we generalize the slope filtration theorem by allowing for ring endomorphisms which do not act as Frobenius lifts on scalars, only on the series variable. This is intended as a prelude to a theory of Frobenius modules in families; we will not develop such a theory here, but see the next section for reasons one might want to do so, from the realm of p-adic Hodge theory. (Note that [**22**] itself generalizes [**21**] in a different direction, replacing the power series rings by somewhat more general objects; we do not treat that generalization here.)

For an alternate perspective on this theorem and some related results in p-adic differential equations and p-adic Hodge theory, we also recommend Colmez's Bourbaki notes [**11**].

0.1. Context. — The slope filtration theorem [**21**, Theorem 6.10] (also exposed in [**22**]) gives a partial classification of Frobenius-semilinear transformations on finite free modules over the Robba ring (a certain ring of univariate formal Laurent series with p-adic coefficients). It is loosely analogous to the eigenspace decomposition of a linear transformation in ordinary linear algebra; it is also closely related to Manin's classification of rational Dieudonné modules.

The slope filtration theorem was originally introduced in the context of Berthelot's rigid cohomology, a p-adic Weil cohomology for varieties in characteristic p. There, one obtains a analogue of the ℓ-adic local monodromy theorem, originally conjectured by Crew [**14**]; this analogue can be used to establish various structural results such as finiteness of cohomology [**23**] and purity in the sense of Deligne [**24**].

The effect of the slope filtration theorem on p-adic Hodge theory has perhaps been even more acute: it enables one to study p-adic Galois representations via their associated (ϕ, Γ)-modules over the Robba ring. This point of view has been put forth chiefly by Berger with striking consequences: he has proved Fontaine's conjecture that de Rham representations are potentially semistable [**4**], and given an alternate proof of the Colmez-Fontaine theorem on admissibility of filtered (ϕ, N)-modules [**5**]. (A useful variant of the latter argument has been given by Kisin [**27**].) More recently Colmez [**13**] used this viewpoint to define a class of *trianguline representations* of a p-adic Galois group; these play an important role in the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ [**12**]. The trianguline representations are also important in the theory of p-adic modular forms, as most local Galois representations attached to overconvergent p-adic modular forms (namely, those of noncritical slope) are trianguline. The p-adic local Langlands correspondence in turn has touched off a flurry of activity, which this introduction is not the right place to summarize; we merely note the resolution of Serre's conjecture by Khare-Wintenberger [**25, 26**], and progress on the Fontaine-Mazur conjecture by Kisin [**28**] and Emerton (in preparation).

In both rigid cohomology and p-adic Hodge theory, one is led to study Frobenius modules in families, i.e., over the Robba ring with coefficients not in a p-adic field but in, say, an affinoid algebra. In either situation, the first step to studying Frobenius modules in families is to pass from a family to a generic point, which on rings amounts to replacing an integral affinoid algebra with a complete field containing it. In the rigid cohomology version of this argument, the resulting field is itself acted on by Frobenius, so the slope filtration theorem as presented in [**21, 22**] is immediately applicable; indeed, the key technique in [**23**] is to extend the application of the local monodromy theorem on the generic point to a large enough subspace of the base space. However, in the p-adic Hodge theory version, one might like to allow "Frobenius" to act in some fashion on the base of the family other than simply a lift of the p-power map; in fact, one natural situation is where the base is not moved at all.

One goal of this paper, and in fact the principal reason for its existence, is to generalize the slope filtration theorem to modules over the Robba ring with an action of a "relative Frobenius", which may do whatever one wishes to coefficients as long as it acts like a Frobenius lift on the series parameter. We hope this will lead to some study of p-adic Hodge theory in families; some of the corresponding analysis in equal characteristics has been initiated by Hartl [**17**], using an equal-characteristic analogue of the slope filtration theorem based on the work of Hartl and Pink [**18**]. In mixed characteristics, Hartl [**16**] has set up part of a corresponding theory, which addresses a conjecture of Rapoport and Zink [**40**] from their work on period spaces for p-divisible groups; results are presently quite fragmentary, but a good theory of (ϕ, Γ)-modules in families may help. Another potential application would be to analysis of the local geometry of the Coleman-Mazur eigencurve [**10**], which parametrizes the Galois representations attached to certain p-adic modular forms, or of higher-dimensional "eigenvarieties" associated to automorphic representations on groups besides GL_2. An initial step in this direction has already been taken by Bellaïche-Chenevier [**3**], who study deformations of trianguline representations; however, this involves only a zero-dimensional base, so they can already apply the usual slope filtration theory after a restriction of scalars. For other questions, e.g., properness, one would want to consider positive-dimensional bases like a punctured disc. In this direction, Berger and Colmez have introduced a theory of étale (ϕ, Γ)-modules associated to p-adic Galois representations in families [**6**], which relativizes some of the results of Cherbonnier-Colmez [**9**] and Berger [**5**] for a single p-adic Galois representation.

0.2. About the results. — For the sake of introduction, we give here a very brief description of what the original slope filtration theorem says, how the main result of this paper extends it, and what novelties in the argument are introduced in this paper. Start with a complete discretely valued field K of mixed characteristics $(0,p)$. Let $\mathscr{R}$ be the ring of formal Laurent series $\sum_{n\in\mathbb{Z}} c_n u^n$ convergent on some annulus with outer radius 1 (but whose inner radius may depend on which series is being considered). Let $\phi_K : K \to K$ be an endomorphism lifting the absolute q-power Frobenius on the residue field of K, for some power q of p, and define a map $\phi : \mathscr{R} \to \mathscr{R}$ by the formula $\phi(\sum c_n u^n) = \sum \phi_K(c_n)\phi(u)^n$, where $\phi(u) - u^q$ has all coefficients of norm less than 1. Let M be a finite free $\mathscr{R}$-module equipped with a ϕ-semilinear map $F : M \to M$ which takes any basis of M to another basis of M (it is enough to check for a single basis). Then [**21**, Theorem 6.10] asserts that M admits an exhaustive filtration whose successive quotients are each pure of some slope (i.e., some power of F times some scalar acts on some basis via an invertible matrix over the subring of $\mathscr{R}$ of series with *integral* coefficients), and the slopes increase as you go up the filtration; moreover, those requirements uniquely characterize the filtration.

As noted earlier, the slope filtration should be thought of as analogous to what one might get from a linear transformation over K by grouping eigenspaces, interpreting the slope of an eigenspace as the valuation of its eigenvalue. One can in fact deduce an analogous such result for semilinear transformations over K, which also follows from the Dieudonné-Manin classification theorem. One might then expect that the slope filtration can be generalized so as to allow *any* isometric action on K, not just a Frobenius lift; that is what is established in this paper (Theorem 1.7.1).

As promised earlier in this introduction, one happy side effect of this generalization is the introduction of some technical simplifications. We give a development of the theory of slopes which does not depend on already having established the Dieudonné-Manin-style classification; this follows up on a suggestion made in [**22**]. We give a much simplified version of the descent argument that deduces the filtration theorem from the DM classification, based on the idea of replacing the Galois descent used previously with faithfully flat descent; this avoids the use of comparison between generic and special Newton polygons, and of some intricate approximation arguments. (In particular, there is no longer any need to deal with finite extensions of the Robba ring, which allows for some notational and expository simplifications.) That substitution creates some flexibility in what we may take as the "extended Robba ring" for the DM classification; here we use a ring made from generalized power series, some of whose properties are a bit more transparent than for the corresponding "big rings" in [**21**] and [**22**].