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THE SIGNATURE PACKAGE ON WITT SPACES

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Abstract. – In this paper we prove a variety of results about the signature operator on Witt spaces. First, we give a parametrix construction for the signature operator on any compact, oriented, stratified pseudomanifold X which satisfies the Witt condition. This construction, which is inductive over the 'depth' of the singularity, is then used to show that the signature operator is essentially self-adjoint and has discrete spectrum of finite multiplicity, so that its index—the analytic signature of X–-is well-defined. This provides an alternate approach to some well-known results due to Cheeger. We then prove some new results. By coupling this parametrix construction to a $C^*_r\Gamma$ Mishchenko bundle associated to any Galois covering of X with covering group Γ, we prove analogues of the same analytic results, from which it follows that one may define an analytic signature index class as an element of the K-theory of $C^*_r\Gamma$. We go on to establish in this setting and for this class the full range of conclusions which sometimes goes by the name of the signature package. In particular, we prove a new and purely topological theorem, asserting the stratified homotopy invariance of the higher signatures of X, defined through the homology L-class of X, whenever the rational assembly map $K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C^*_r\Gamma) \otimes \mathbb{Q}$ is injective.

Résumé. – Dans cet article nous prouvons plusieurs résultats pour l'opérateur de la signature sur un espace de Witt X compact orienté quelconque. Nous construisons une paramétrix de l'opérateur de la signature de X en raisonnant par récurrence sur la profondeur de X et en utilisant une analyse très fine de l'opérateur normal (près d'une strate). Ceci nous permet de montrer que le domaine maximal de l'opérateur de la signature est compactement inclus dans l'espace L^2 correspondant. On peut alors (re)démontrer que l'opérateur de la signature est essentiellement self-adjoint et a un spectre L^2 discret de multiplicité finie de sorte que son indice est bien défini. Nous donnons donc une nouvelle démonstration de certains résultats dus à Jeff Cheeger. Nous considérons ensuite le cas où X est muni d'un revêtement galoisien de groupe Γ. Nous utilisons alors nos constructions pour définir la classe d'indice de signature analytique à valeurs dans le groupe de K-théorie $K_*(C^*_r\Gamma)$. Nous généralisons dans cette situation singulière la plupart des résultats connus dans le cas où X est lisse. C'est ce qu'on appelle le « forfait signature ». En particulier, nous prouvons un nouveau théorème, purement topologique, qui permet de prouver l'invariance par homotopie stratifiée des hautes signatures de X (définies à l'aide de la $L-$classe homologique de X) pourvu que l'application d'assemblement rationnelle $K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C^*_r\Gamma) \otimes \mathbb{Q}$ soit injective.

1. Introduction

Let X be an orientable closed compact Riemannian manifold with fundamental group Γ. Let X' be a Galois Γ-covering and $r : X \to B\Gamma$ a classifying map for X'. The *signature package* for the pair $(X, r: X \to B\Gamma)$ refers to the following collection of results:

1. the signature operator with values in the Mishchenko bundle $r^*E\Gamma \times_\Gamma C^*_r\Gamma$ defines a signature index class $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}}) \in K_*(C^*_r\Gamma)$, $* \equiv \dim X \pmod 2$;
2. the signature index class is a bordism invariant; more precisely it defines a group homomorphism $\Omega^{\mathrm{SO}}_*(B\Gamma) \to K_*(C^*_r\Gamma)$;
3. the signature index class is a homotopy invariant;
4. there is a K-homology signature class $[\eth_{\mathrm{sign}}] \in K_*(X)$ whose Chern character is, rationally, the Poincaré dual of the L-Class;
5. the assembly map $\beta : K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ sends the class $r_*[\eth_{\mathrm{sign}}]$ into $\mathrm{Ind}(\widetilde{\eth}_{\mathrm{sign}})$;
6. if the assembly map is rationally injective, one can deduce from (1) - (5) that the Novikov higher signatures
$$\{\langle L(X) \cup r^*\alpha, [X]\rangle,\ \alpha \in H^*(B\Gamma, \mathbb{Q})\}$$
are homotopy invariant.

We call this list of results, together with the following item, the *full* signature package:

(7) there is a (C^*-algebraic) symmetric signature $\sigma_{C^*_r\Gamma}(X, r) \in K_*(C^*_r\Gamma)$, which is topologically defined, a bordism invariant $\sigma_{C^*_r\Gamma} : \Omega^{\mathrm{SO}}_*(B\Gamma) \to K_*(C^*_r\Gamma)$ and, in addition, is equal to the signature index class.

For history and background see [16] [51] and for a survey we refer to [30].

The main goal of this paper is to formulate and establish the signature package for a class of stratified pseudomanifolds known as Witt spaces. In particular, we prove by analytic methods a new and purely topological result concerning the stratified homotopy invariance of suitably defined higher signatures under an injectivity assumption on the assembly map for the group Γ.

The origins of the signature package on a closed oriented manifold X can be traced back to the Atiyah-Singer proof of the signature formula of Hirzebruch, $\sigma_{\mathrm{top}}(X) = \mathcal{L}(X) := \langle L(X), [X]\rangle$. In this proof the central object is the Fredholm index of the signature operator which is proved to be simultaneously equal to the topological signature of the manifold $\sigma_{\mathrm{top}}(X)$ and to its L-genus $\mathcal{L}(X)$:
$$\sigma_{\mathrm{top}}(X) = \mathrm{ind}(\eth_{\mathrm{sign}}) = \mathcal{L}(X)\,.$$
The idea of using index theory to investigate topological properties of X received new impetus through the seminal work of Lusztig, who used the family index theorem of Atiyah-Singer in order to establish the Novikov conjecture on the homotopy invariance of the higher signatures of X when $\pi_1(X) = \mathbb{Z}^k$. Most of the signature package as formulated here can be seen as a noncommutative version of the results of Lusztig. Crucial in the formulation and proof of the signature package are the following issues:

– the Poincaré duality property for the (co)homology of X and more generally, the Algebraic Poincaré Complex structure of its (co)chain complex;
– the possibility of defining bordism groups $\Omega^{\mathrm{SO}}(T)$, T a topological space, with cycles given by closed oriented manifolds endowed with a reference map to T;

– an elliptic theory which allows one to establish the analytic properties of $\eth_{\rm sign}$ and then connect them to the topological properties of X;
– the possibility of extending this elliptic theory to signature operators twisted by a bundle of finitely generated projective A-modules, where A is a C^*-algebra. The prototype is the signature operator $\widetilde{\eth}_{\rm sign}$ twisted by the Mishchenko bundle $r^*E\Gamma \times_\Gamma C^*_r\Gamma$.

Once one moves from closed oriented manifold to stratified pseudomanifolds, many of these issues need careful reformulation and substantially more care. First, it is well-known that Poincaré duality fails on a general stratified pseudomanifold $\widehat{X}$. Next, the bordism group $\Omega^{\rm pseudo}(T)$, the cycles of which are *arbitrary* stratified pseudomanifolds endowed with a reference map to T, is not the right one; indeed, as explained in [4], the coefficients of such a theory, $\Omega^{\rm pseudo}(\text{point})$, are trivial. Finally, the analytic properties of the signature operator on the regular part of a stratified pseudomanifold endowed with an 'incomplete iterated edge metric' (which is a particularly simple and natural type of metric that can be constructed on such a space) are much more delicate than in the closed case. In particular, this operator may not even be essentially self-adjoint, and the possibility of numerous distinct self-adjoint extensions complicates the possible connections to topology.

The first problem has been tackled by Goresky and MacPherson in the topological setting [20] [21] and by Cheeger in the analytic setting [11] [12] (at least for the particular subclass of stratified pseudomanifolds we consider below). The search for a cohomology theory on such spaces with some vestiges of Poincaré duality led Goresky and MacPherson to their discovery of intersection (co)homology groups, $IH^*_p(\widehat{X}, \mathbb{Q})$, where p is a 'perversity function', and to the existence of a perfect pairing

$$IH^*_p(\widehat{X}, \mathbb{Q}) \times IH^*_q(\widehat{X}, \mathbb{Q}) \to \mathbb{Q}$$

where p and q are complementary perversities. Notice that we still do not obtain a signature unless the perversities can be chosen the same, i.e. unless there is a perfect pairing

$$IH^*_m(\widehat{X}, \mathbb{Q}) \times IH^*_m(\widehat{X}, \mathbb{Q}) \to \mathbb{Q}$$

for some perversity function m. Witt spaces constitute a subclass of stratified pseudomanifolds for which all of these difficulties can be overcome.

A stratified pseudomanifold $\widehat{X}$ is a Witt space if any even-dimensional link L satisfies $IH^{\dim L/2}_m(L, \mathbb{Q}) = 0$, where m is the upper-middle perversity function. Examples of Witt spaces include any singular projective variety over $\mathbb{C}$. We list some particularly interesting properties of Witt spaces:

– the upper-middle and lower-middle perversity functions define the same intersection cohomology groups, which are then denoted by $IH^*_m(\widehat{X})$;
– there is a perfect pairing

$$IH^*_m(\widehat{X}, \mathbb{Q}) \times IH^*_m(\widehat{X}, \mathbb{Q}) \to \mathbb{Q}\ ;$$

in particular, there is a well defined intersection cohomology signature;
– there are well-defined and nontrivial Witt bordism groups $\Omega^{\rm Witt}(T)$ (for example, these are rationally isomorphic to the connected version of KO-homology, $ko(T) \otimes_{\mathbb{Z}} \mathbb{Q}$);
– there is a class of Riemannian metrics on the regular part of $\widehat{X}$ for which
  - the signature operator is essentially self-adjoint

- its unique self-adjoint extension has discrete spectrum of finite multiplicity
- there is a de Rham-Hodge theorem, connecting the Hodge cohomology, the L^2-cohomology and the intersection cohomology $IH^*_m(\widehat{X}, \mathbb{C})$.

The topological results here are due to Goresky-MacPherson and Siegel. The analytic results are due initially to Cheeger, though there is much further work in this area, see, for example, [9], [39], [27], [54]. Cheeger's results on the signature operator are based on a careful analysis of the heat kernel of the associated Laplacian.

We have a number of goals in this article:

- we give a new treatment of Cheeger's result on the signature operator based on the methods of geometric microlocal analysis;
- this approach is then adapted to the signature operator $\widetilde{\eth}_{\rm sign}$ with value in the Mishchenko bundle $r^*E\Gamma \times_\Gamma C^*_r\Gamma$;
- we carefully analyze the resulting index class, with particular emphasis on its stability property;
- we collect this analytic information and establish the whole range of results encompassed by the signature package on Witt spaces. In particular, we prove a *Novikov conjecture on Witt spaces* whenever the assembly map for the fundamental group is rationally injective. We note again that this is a new and purely topological result.

This article is divided into three parts. In the first one, we give a detailed account of the resolution, through a series of blowups, of an arbitrary stratified pseudomanifold (not necessarily satisfying the Witt condition) to a manifold with corners. This has been studied in the past, most notably by Verona [59]; the novelty in our treatment is the introduction of iterated fibration structures, a notion due to Melrose, as an extra structure on the boundary faces of the resolved manifold with corners. We also show that a manifold with corners with an iterated fibration structure can be blown down to a stratified pseudomanifold. In other words, the classes of stratified pseudomanifolds and of manifolds with corners with iterated fibration structure are equivalent. Much of this material is based on unpublished work by Richard Melrose, and we are grateful to him for letting us use and develop these ideas here. We then describe the *(incomplete) iterated edge metrics*, which are the simplest type of incomplete metrics adapted to this class of singular space. We show in particular that the space of such metrics is nonempty and path-connected. We also consider, for any such metric, certain conformally related complete, and 'partially complete' metrics used in the ensuing analysis.

The second part of this article focuses on the analysis of natural elliptic operators, specifically, the de Rham and signature operators, associated to incomplete iterated edge metrics. Our methods are drawn from geometric microlocal analysis. Indeed, in the case of simply stratified spaces, with only one singular stratum, there is a very detailed pseudodifferential theory [41] which can be used for problems of this type, and in the even simpler case of manifolds with isolated conic singularities, one may use the somewhat simpler b-calculus of Melrose, see [44]. In either of these cases, a crucial step is to consider the de Rham or signature operator associated to an incomplete edge or conic metric as a singular factor multiplying an elliptic operator in the edge or b-calculus, and then to study this latter, auxiliary, operator