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LEFSCHETZ FOR LOCAL PICARD GROUPS

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ABSTRACT. – We prove a strengthening of the Grothendieck-Lefschetz hyperplane theorem for local Picard groups conjectured by Kollár. Our approach, which relies on acyclicity results for absolute integral closures, also leads to a restriction theorem for higher rank bundles on projective varieties in positive characteristic.

RÉSUMÉ. – Nous prouvons un renforcement du théorème de l'hyperplan de Grothendieck-Lefschetz pour les groupes locaux de Picard conjecturés par Kollár. Notre approche, qui s'appuie sur des résultats en fermetures absolues, conduit également à un théorème de restriction pour les faisceaux de rang supérieur sur les variétés projectives en caractéristique positive.

A classical theorem of Lefschetz asserts that non-trivial line bundles on a smooth projective variety of dimension $\geq$ 3 remain non-trivial upon restriction to an ample divisor, and plays a fundamental role in understanding the topology of algebraic varieties. In [6], Grothendieck recast this result in more general terms using the machinery of formal geometry and deformation theory, and also stated a local version. With a view towards moduli of higher dimensional varieties, especially the deformation theory of log canonical singularities, Kollár recently conjectured [15] that Grothendieck's local formulation remains true under weaker hypotheses than those imposed in [6]. Our goal in this paper is to prove Kollár's conjecture for rings containing a field.

Statement of results

Let $(A, \mathfrak{m})$ be an excellent normal local ring containing a field. Fix some $0 \neq f \in \mathfrak{m}$. Let $V = \mathrm{Spec}(A) - \{\mathfrak{m}\}$, and $V_0 = \mathrm{Spec}(A/f) - \{\mathfrak{m}\}$. The following result is the key theorem in this paper; it solves [15, Problem 1.3] completely, and [15, Problem 1.2] in characteristic 0:

THEOREM 0.1. – *Assume* $\dim(A) \geq 4$. *The restriction map* $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$ *is:*

1. *injective if* $\mathrm{depth}_{\mathfrak{m}}(A/f) \geq 2$ *and A has characteristic* 0*;*
2. *injective up to* p^{∞}*-torsion if A has characteristic* $p > 0$.

This result is sharp: surjectivity fails in general, while injectivity fails in general if $\dim(A) \leq 3$, in characteristic 0 if $\mathrm{depth}_{\mathfrak{m}}(A/f) < 2$, and in characteristic p if one includes p-torsion. Theorem 0.1 leads to a fibral criterion for a Weil divisor to be Cartier in a family, see Theorem 1.30. A stronger analogue of Theorem 0.1, including the mixed characteristic case, is due to Grothendieck [6, Expose XI] under the stronger condition $\mathrm{depth}_{\mathfrak{m}}(A/f) \geq 3$; complex analytic variants of Grothendieck's theorem are proven in [7], while topological analogues are discussed in [9]. Without this depth constraint, a previously known case of Theorem 0.1 was when A has log canonical singularities in characteristic 0, and $\{\mathfrak{m}\} \subset \mathrm{Spec}(A)$ is not an lc center (see [15, Theorem 19]).

Our approach to Theorem 0.1 relies on formal geometry over absolute integral closures [2, 11], and applies to higher rank bundles as well as projective varieties. This technique then leads to a short proof of the following result:

THEOREM 0.2. – *Let X be a normal projective variety of dimension $d \geq 3$ over an algebraically closed field of characteristic $p > 0$. If a vector bundle E on X is trivial over an ample divisor, then $(\mathrm{Frob}_X^e)^* E \simeq \mathcal{O}_X^{\oplus r}$ for $e \gg 0$.*

The numerical version of Theorem 0.2 for line bundles is due to Kleiman [13, Corollary 2, page 305]. The non-numerical version of the rank 1 case, with stronger assumptions on the singularities, is studied in [8]. This result may also be deduced from the boundedness [16] of semistable sheaves. We do not know the correct characteristic 0 analogue of this result.

An outline of the proof

Both theorems are similar in spirit, so we only discuss Theorem 0.1 here. We first prove the characteristic p result, and then deduce the characteristic 0 one by reduction modulo p and an approximation argument; the reduction necessitates the (unavoidable) depth assumption in characteristic 0. The characteristic p proof follows Grothendieck's strategy of decoupling the problem into two pieces: one in formal f-adic geometry, and the other an algebraization question. Our main new idea is to replace (thanks entirely to the Hochster-Huneke vanishing theorem [11]) our ring A with a very large extension $\overline{A}$ with better depth properties; Grothendieck's deformation-theoretic approach then immediately solves the formal geometry problem over $\overline{A}$. Next, we algebraize the solution over $\overline{A}$ by algebraically approximating formal sections of line bundles; the key here is to identify the cohomology of the formal completion of a scheme as the *derived* completion of the cohomology of the original scheme, i.e., a weak analogue of the formal functions theorem devoid of the usual finiteness constraints. Finally, we descend from $\overline{A}$ to A; this step is trivial in our context, but witnesses the torsion in the kernel.

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1. Local Picard groups

The goal of this section is to prove Theorem 0.1. In §1.1, we study formal geometry along a divisor on a (punctured) local scheme abstractly, and establish certain criteria for restriction map on Picard groups to be injective. These are applied in §1.2 to prove the characteristic p part of Theorem 0.1. Using the principle of "reduction modulo p" and a standard approximation argument (sketched in §1.4), we prove the characteristic 0 part of Theorem 0.1 in §1.3. The afore-mentioned fibral criterion is recorded in §1.5. Finally, in §1.6, we give examples illustrating the necessity of the assumptions in Theorem 0.1.

1.1. Formal geometry over a punctured local scheme

We establish some notation that will be used in this section.

Notation 1.1. – Let $(A, \mathfrak{m})$ be a local ring, and fix a regular element $f \in \mathfrak{m}$. Let $X = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(A) - \{\mathfrak{m}\}$. For an X-scheme Y, write Y_n for the reduction of Y modulo f^{n+1}, and $\widehat{Y}$ for the formal completion[(1)] of Y along Y_0. Let $\mathrm{Vect}(Y)$ be the category of vector bundles (i.e., finite rank locally free sheaves) on Y, and write $\mathrm{Pic}(Y)$ and $\underline{\mathrm{Pic}}(Y)$ for the set and groupoid of line bundles respectively. Set $\underline{\mathrm{Pic}}(\widehat{Y}) := \lim \underline{\mathrm{Pic}}(Y_n)$ (where the limit is in the sense of groupoids), and $\mathrm{Pic}(\widehat{Y}) := \pi_0(\underline{\mathrm{Pic}}(\widehat{Y}))$. For any A-module M with associated quasi-coherent sheaf $\widetilde{M}$ on $\mathrm{Spec}(A)$, we *define* $H^i_{\mathfrak{m}}(M)$ as cohomology supported along $\{\mathfrak{m}\} \subset X$ of $\widetilde{M}$, i.e., as the ith cohomology of the complex $\mathrm{R}\Gamma_{\mathfrak{m}}(M)$ defined as the homotopy-kernel of the map $\mathrm{R}\Gamma(\mathrm{Spec}(A), \widetilde{M}) \to \mathrm{R}\Gamma(V, \widetilde{M})$.

We will use formal schemes associated to certain non-Noetherian X-schemes later in this paper. Rather than developing the general theory of such schemes, we simply define the concept that will be most relevant: cohomology.

Definition 1.2. – Fix an X-scheme Y. For $F \in D(\mathcal{O}_Y)$, set $\widehat{F} := \mathrm{R}\lim(F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n})$; we view $\widehat{F}$ as an $\mathcal{O}_{\widehat{Y}} := \lim_n \mathcal{O}_{Y_n}$-complex on $|\widehat{Y}| := Y_0$, so $\mathrm{R}\Gamma(\widehat{Y}, \widehat{F}) := \mathrm{R}\Gamma(Y_0, \widehat{F}) \simeq \mathrm{R}\lim \mathrm{R}\Gamma(Y_0, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n})$.

The following two examples help explain the meaning of this definition:

Example 1.3. – If F is a quasicoherent $\mathcal{O}_X$-module associated to an A-module M, then $\mathrm{R}\Gamma(\widehat{X}, \widehat{F}) \simeq \mathrm{R}\lim(M \otimes^L_A A/(f^n))$. In particular, if M is A-flat, then $\mathrm{R}\Gamma(\widehat{X}, \widehat{F})$ is the f-adic completion of M in the usual sense. Note that if M is not A-flat, then $\mathrm{R}\Gamma(\widehat{X}, \widehat{F})$ could have cohomology in negative degrees.

Example 1.4. – Fix a quasicoherent flat $\mathcal{O}_V$-module F, assumed to be obtained from an A-module M via localization. Then $\mathrm{R}\Gamma(\widehat{V}, \widehat{F})$ is computed as follows. Fix an ideal $(g_1, \ldots, g_r) \subset A$ with $V(g_1, \ldots, g_r) = \{\mathfrak{m}\}$ set-theoretically (assumed to exist). Let $C(M; g_1, \ldots, g_r) := \bigotimes_{i=1}^r \left(M \xrightarrow{1} M_{g_i}\right)$ be the displayed Cech complex, and let $K(M)$ be the cone of the natural map $C(M; g_1, \ldots, g_r) \to M$. Then the (termwise) f-adic completion of K computes $\mathrm{R}\Gamma(\widehat{V}, \widehat{F})$. To see this, observe first that $K(M)/f^n K(M)$ computes

[(1)] The formal scheme $\widehat{Y}$ is used as a purely linguistic device to talk about compatible systems of sheaves on each Y_n, and not in a deeper manner.

$\mathrm{R}\Gamma(V_n, F \otimes^L_{\mathcal{O}_V} \mathcal{O}_{V_n})$. It follows that the term-wise f-adic completion of K computes $\mathrm{R}\lim \mathrm{R}\Gamma(V_n, F \otimes_{\mathcal{O}_V} \mathcal{O}_{V_n}) \simeq \mathrm{R}\Gamma(\widehat{V}, \widehat{F})$.

The derived completion functor $K \mapsto \mathrm{R}\lim(K \otimes^L_A A/f^n)$ already appears implicitly in the above definition. To access its values, recall the following definition:

Definition 1.5. – Given an A-module M, we define the *f-adic Tate module* as $T_f(M) := \lim M[f^n]$ with transition maps given by powers of f; note that $T_f(M) = 0$ if $f^N \cdot M = 0$ for some $N > 0$.

The Tate module leads to the second of the following two descriptions of the cohomology of a formal completion:

Lemma 1.6. – *Let Y be an X-scheme such that $\mathcal{O}_Y$ has bounded f^∞-torsion. For $F \in D(\mathcal{O}_Y)$, there are exact sequences*

$$1 \to \mathrm{R}^1 \lim H^{i-1}(Y_n, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n}) \to H^i(\widehat{Y}, \widehat{F}) \to \lim H^i(Y, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n}) \to 1,$$

and

$$1 \to \lim H^i(Y, F)/f^n \to H^i(\widehat{Y}, \widehat{F}) \to T_f(H^{i+1}(Y, F)) \to 1.$$

Proof. – We first give a proof when $\mathcal{O}_Y$ has no f-torsion (which will be the only relevant case in the sequel). The first sequence is then obtained from the formula

$$\mathrm{R}\Gamma(\widehat{Y}, \widehat{F}) \simeq \mathrm{R}\lim \mathrm{R}\Gamma(Y, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n})$$

and Milnor's exact sequence for $\mathrm{R}\lim$ (see [18]). Applying the projection formula (since A/f^n is A-perfect) to the above gives

$$\mathrm{R}\Gamma(\widehat{Y}, \widehat{F}) \simeq \mathrm{R}\lim \big(\mathrm{R}\Gamma(Y, F) \otimes^L_A A/f^n\big).$$

The second sequence is now obtained by applying the derived f-adic completion functor $\mathrm{R}\lim(- \otimes^L_A A/f^n)$ to the canonical filtration on $\mathrm{R}\Gamma(Y, F)$, which proves the claim. In general, the boundedness of f-torsion in $\mathcal{O}_Y$ shows that the map $\{\mathcal{O}_Y \xrightarrow{f^n} \mathcal{O}_Y\} \to \{\mathcal{O}_{Y_n}\}$ of projective systems is a (strict) pro-isomorphism, and hence $\{F \xrightarrow{f^n} F\} \to \{F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n}\}$ is also a pro-isomorphism. Now the previous argument applies. ☐

The following conditions on the data (A, f) will be assumed throughout this subsection; we do *not* assume A is Noetherian as this will not be true in applications.

Assumption 1.7. – Assume that the data from Notation 1.1 satisfies the following:

- X is integral, i.e., A is a domain;
- $j : V \hookrightarrow X$ is a quasi-compact open immersion, i.e., $\mathfrak{m}$ is the radical of a finitely generated ideal;
- $H^0(V, \mathcal{O}_V)$ is a finite A-module;
- $f^N \cdot H^1(V, \mathcal{O}_V) = 0$ for $N \gg 0$.