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CARTAN SUBALGEBRAS OF AMALGAMATED FREE PRODUCT II_1 FACTORS

BY ADRIAN IOANA
WITH AN APPENDIX BY ADRIAN IOANA AND STEFAAN VAES

Dedicated to Sorin Popa

ABSTRACT. – We study Cartan subalgebras in the context of amalgamated free product II_1 factors and obtain several uniqueness and non-existence results. We prove that if Γ belongs to a large class of amalgamated free product groups (which contains the free product of any two infinite groups) then any II_1 factor $L^\infty(X) \rtimes \Gamma$ arising from a free ergodic probability measure preserving action of Γ has a unique Cartan subalgebra, up to unitary conjugacy. We also prove that if $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ is the free product of any two non-hyperfinite countable ergodic probability measure preserving equivalence relations, then the II_1 factor $L(\mathcal{R})$ has a unique Cartan subalgebra, up to unitary conjugacy. Finally, we show that the free product $M = M_1 * M_2$ of any two II_1 factors does not have a Cartan subalgebra. More generally, we prove that if $A \subset M$ is a diffuse amenable von Neumann subalgebra and $P \subset M$ denotes the algebra generated by its normalizer, then either P is amenable, or a corner of P can be unitarily conjugate into M_1 or M_2.

RÉSUMÉ. – Nous étudions les sous-algèbres de Cartan dans le contexte du produit amalgamé de facteurs de type II_1 et nous obtenons plusieurs résultats d'unicité et de non-existence. Nous démontrons que, si Γ appartient à une grande classe de produits amalgamés de groupes (qui contient le produit libre de deux groupes infinis), alors tout facteur de type II_1 associé à une action libre ergodique de Γ a une sous-algèbre de Cartan unique, à conjugaison unitaire. Nous démontrons aussi que, si $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ est le produit libre de toute relation d'équivalence ergodique non-hyperfinie dénombrable, alors le facteur de type II_1 $L(\mathcal{R})$ a une sous-algèbre de Cartan unique, à conjugaison unitaire. Enfin, nous démontrons que le produit libre $M = M_1 * M_2$ de tout facteur de type II_1 n'a pas de sous-algèbre de Cartan. Plus généralement, nous démontrons que, si $A \subset M$ est une sous-algèbre de von Neumann amenable et non-atomique et si $P \subset M$ désigne l'algèbre engendrée par son normalisateur, alors soit P est amenable, soit un coin de P peut être unitairement conjugué dans M_1 ou M_2.

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1. Introduction

A *Cartan subalgebra* of a II_1 factor M is a maximal abelian von Neumann subalgebra A whose normalizer generates M. The study of Cartan subalgebras plays a central role in the classification of II_1 factors arising from probability measure preserving (pmp) actions. If $\Gamma \curvearrowright (X,\mu)$ is a free ergodic pmp action of a countable group Γ, then the *group measure space* II_1 factor $L^\infty(X)\rtimes\Gamma$ [38] contains $L^\infty(X)$ as a Cartan subalgebra. In order to classify $L^\infty(X)\rtimes\Gamma$ in terms of the action $\Gamma \curvearrowright X$, one would ideally aim to show that $L^\infty(X)$ is its unique Cartan subalgebra (up to conjugation by an automorphism). Proving that certain classes of group measure space II_1 factors have a unique Cartan subalgebra is useful because it reduces their classification, up to isomorphism, to the classification of the corresponding actions, up to orbit equivalence. Indeed, following [58, 15], two free ergodic pmp actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are *orbit equivalent* if and only if there exists an isomorphism $\theta : L^\infty(X)\rtimes\Gamma \to L^\infty(Y)\rtimes\Lambda$ such that $\theta(L^\infty(X)) = L^\infty(Y)$.

In the case of II_1 factors coming from actions of amenable groups, both the classification and uniqueness of Cartan problems have been completely settled since the early 1980's. A celebrated theorem of A. Connes [67] asserts that all II_1 factors arising from free ergodic pmp actions of infinite amenable groups are isomorphic to the hyperfinite II_1 factor, R. Additionally, [13] shows that any two Cartan subalgebras of R are conjugate by an automorphism of R.

For a long time, however, the questions of classification and uniqueness of Cartan subalgebras for II_1 factors associated with actions of non-amenable groups, were considered intractable. During the last decade, S. Popa's *deformation/rigidity* theory has led to spectacular progress in the classification of group measure space II_1 factors (see the surveys [49, 62, 30]). This was in part made possible by several results providing classes of group measure space II_1 factors that have a unique Cartan subalgebra, up to unitary conjugacy. The first such classes were obtained by N. Ozawa and S. Popa in their breakthrough work [41, 42]. They showed that II_1 factors $L^\infty(X)\rtimes\Gamma$ associated with free ergodic *profinite* actions of free groups $\Gamma=\mathbb{F}_n$ and their direct products $\Gamma=\mathbb{F}_{n_1}\times\mathbb{F}_{n_2}\times\cdots\times\mathbb{F}_{n_k}$ have a unique Cartan subalgebra, up to unitary conjugacy. Recently, this result has been extended to profinite actions of hyperbolic groups [10] and of direct products of hyperbolic groups [11]. The proofs of these results rely both on the fact that free groups (and, more generally, hyperbolic groups, see [39, 40]) are *weakly amenable* and that the actions are profinite.

In a very recent breakthrough, S. Popa and S. Vaes succeeded in removing the profiniteness assumption on the action and obtained wide-ranging unique Cartan subalgebra results. They proved that if Γ is either a weakly amenable group with $\beta_1^{(2)}(\Gamma)>0$ [55] or a hyperbolic group [56] (or a direct product of groups in one of these classes), then II_1 factors $L^\infty(X)\rtimes\Gamma$ arising from *arbitrary* free ergodic pmp actions of Γ have a unique Cartan subalgebra, up to unitary conjugacy. Following [55, Definition 1.4], such groups Γ, whose every action gives rise to a II_1 factor with a unique Cartan subalgebra, are called *$\mathscr{C}$-rigid* (Cartan rigid).

In this paper we study Cartan subalgebras of tracial amalgamated free product von Neumann algebras $M = M_1 *_B M_2$ (see [46, 66] for the definition). Our methods are best suited to the case when $M = L^\infty(X)\rtimes\Gamma$ comes from an action of an amalgamated free product group $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$. In this context, by imposing that the inclusion $\Lambda < \Gamma$ satisfies a weak malnormality condition [53], we prove that $L^\infty(X)$ is the unique Cartan subalgebra of M, up to unitary conjugacy, for *any* free ergodic pmp action $\Gamma \curvearrowright X$.

THEOREM 1.1. – *Let* $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ *be an amalgamated free product group such that* $[\Gamma_1 : \Lambda] \geqslant 2$ *and* $[\Gamma_2 : \Lambda] \geqslant 3$. *Assume that there exist* $g_1, g_2, \dots, g_n \in \Gamma$ *such that* $\bigcap_{i=1}^n g_i \Lambda g_i^{-1}$ *is finite. Let* $\Gamma \curvearrowright (X, \mu)$ *be any free ergodic pmp action of* Γ *on a standard probability space* (X, μ).

Then the II_1 *factor* $M = L^\infty(X) \rtimes \Gamma$ *has a unique Cartan subalgebra, up to unitary conjugacy.*

Moreover, the same holds if Γ *is replaced with a direct product of finitely many such groups* Γ.

This result provides the first examples of $\mathscr{C}$-rigid groups Γ that are not weakly amenable (take e.g., $\Gamma = SL_3(\mathbb{Z}) * \Sigma$, where Σ is any non-trivial countable group).

Theorem 1.1 generalizes and strengthens the main result of [53]. Indeed, in the above setting, assume further that Λ is amenable and that Γ_2 contains either a non-amenable subgroup with the relative property (T) or two non-amenable commuting subgroups. [53, Theorem 1.1] then asserts that M has a unique *group measure space* Cartan subalgebra.

Theorem 1.1 provides strong supporting evidence for a general conjecture which predicts that any group Γ with positive first ℓ^2-Betti number, $\beta_1^{(2)}(\Gamma) > 0$, is $\mathscr{C}$-rigid. Thus, it implies that the free product $\Gamma = \Gamma_1 * \Gamma_2$ of any two countable groups satisfying $|\Gamma_1| \geqslant 2$ and $|\Gamma_2| \geqslant 3$, is $\mathscr{C}$-rigid.

Recently, there have been several results offering positive evidence for this conjecture. Firstly, it was shown in [53] that if $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1 is a property (T) group and Γ_2 is a non-trivial group, then any II_1 factor $L^\infty(X) \rtimes \Gamma$ associated with a free ergodic pmp action of Γ has a unique group measure space Cartan subalgebra, up to unitary conjugacy (see also [16, 24]). Secondly, the same has been proven in [9] under the assumption that $\beta_1^{(2)}(\Gamma) > 0$ and Γ admits a non-amenable subgroup with the relative property (T). For a common generalization of the last two results, see [63]. Thirdly, we proved that if $\beta_1^{(2)}(\Gamma) > 0$, then $L^\infty(X) \rtimes \Gamma$ has a unique group measure space Cartan subalgebra whenever the action $\Gamma \curvearrowright (X, \mu)$ is either rigid [29] or compact [28]. As already mentioned above, the conjecture has been very recently established in full generality for weakly amenable groups Γ with $\beta_1^{(2)}(\Gamma) > 0$ in [55].

As a consequence of Theorem 1.1 we obtain a new family of W^*-superrigid actions. Recall that a free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ is called W^*-*superrigid* if whenever $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$, for some free ergodic pmp action $\Lambda \curvearrowright (Y, \nu)$, the groups Γ and Λ are isomorphic, and their actions are conjugate. The existence of virtually W^*-superrigid actions was proven in [43]. The first concrete families of W^*-superrigid actions were found in [53] where it was shown for instance that Bernoulli actions of many amalgamated free product groups have this property. In [27] we proved that Bernoulli actions of icc property (T) groups are W^*-superrigid. By combining Theorem 1.1 with the cocycle superrigidity theorem [51] we derive the following.