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DYNAMICAL ZETA FUNCTIONS FOR ANOSOV FLOWS VIA MICROLOCAL ANALYSIS

BY SEMYON DYATLOV AND MACIEJ ZWORSKI

ABSTRACT. – The purpose of this paper is to give a short microlocal proof of the meromorphic continuation of the Ruelle zeta function for C^∞ Anosov flows. More general results have been recently proved by Giulietti-Liverani-Pollicott [13] but our approach is different and is based on the study of the generator of the flow as a semiclassical differential operator.

RÉSUMÉ. – Cet article donne une courte preuve microlocale du prolongement méromorphe de la fonction zêta de Ruelle pour les flots d'Anosov lisses. Des résultats plus généraux ont été récemment obtenus par Giulietti-Liverani-Pollicott [13] mais notre approche est différente et se base sur l'étude du générateur du flot, que l'on considère comme un opérateur pseudodifférentiel semi-classique.

The purpose of this article is to provide a short microlocal proof of the meromorphic continuation of the Ruelle zeta function for C^∞ Anosov flows on compact manifolds:

THEOREM. – *Suppose X is a compact manifold and $\varphi_t : X \to X$ is a C^∞ Anosov flow with orientable stable and unstable bundles. Let $\{\gamma^\sharp\}$ denote the set of primitive orbits of φ_t, with $T_\gamma^\sharp$ their periods. Then the Ruelle zeta function,*

$$\zeta_{\mathrm{R}}(\lambda) = \prod_{\gamma^\sharp} (1 - e^{i\lambda T_\gamma^\sharp}),$$

which converges for $\operatorname{Im} \lambda \gg 1$ *has a meromorphic continuation to* $\mathbb{C}$.

In fact the proof applies to any Anosov flow for which linearized Poincaré maps $\mathcal{P}_\gamma$ for closed orbits γ satisfy

$$|\det(I - \mathcal{P}_\gamma)| = (-1)^q \det(I - \mathcal{P}_\gamma), \quad \text{with } q \text{ independent of } \gamma. \tag{1.1}$$

A class of examples is provided by $X = S^*M$ where M is a compact orientable negatively curved manifold with φ_t the geodesic flow—see [13, Lemma B.1]. For methods which can be used to eliminate the orientability assumptions, see [13, Appendix B].

The meromorphic continuation of ζ_{R} was conjectured by Smale [33] and in greater generality it was proved very recently by Giulietti, Liverani, and Pollicott [13]. Another recent perspective on dynamical zeta functions in the contact case has been provided by Faure and

Tsujii [10, 11]. Our motivation and proof are however different from those of [13]: we were investigating trace formulæ for Pollicott-Ruelle resonances [28, 30] which give some lower bounds on their counting function. Sharp upper bounds were given recently in [4, 9].

To explain the trace formula for resonances suppose first that $X = S^*\Gamma\backslash\mathbb{H}^2$ is a compact Riemann surface. Then the Selberg trace formula combined with the Guillemin trace formula [17] gives

$$\sum_{\mu\in\operatorname{Res}(P)} e^{-i\mu t} = \sum_{\gamma} \frac{T_\gamma^\# \delta(t-T_\gamma)}{|\det(I-\mathcal{P}_\gamma)|}, \quad t>0, \tag{1.2}$$

see [24] for an accessible presentation in the physics literature and [6] for the case of higher dimensions. On the left hand side $\operatorname{Res}(P)$ is the set of resonances of $P=-iV$ where V is the generator of the flow,

$$\operatorname{Res}(P) = \left\{\mu_{j,k} = \lambda_j - i(k+\tfrac{1}{2}),\ j,k\in\mathbb{N}\right\},$$

where λ_j's are the zeros of the Selberg zeta function included according to their multiplicities. On the right hand side γ's are periodic orbits, $\mathcal{P}_\gamma$ is the linearized Poincaré map, T_γ is the period of γ, and $T_\gamma^\#$ is the primitive period.

The point of view of Faure-Sjöstrand [9] stresses the analogy between analysis of the propagator $\varphi_{-t}^* = e^{-itP}$ with scattering theory for elliptic operators on non-compact manifolds: for flows, the fiber infinity of T^*X is the analogue of spatial infinity for scattering on non-compact manifolds. Melrose's Poisson formula for resonances valid for Euclidean infinities [26, 32, 36] and some hyperbolic infinities [18] suggests that (1.2) should be valid for general Anosov flows but that seems to be unknown.

In general, the validity of (1.2) follows from the finite order (as an entire function) of the analytic continuation of

$$\zeta_1(\lambda) := \exp\left(-\sum_{\gamma}\frac{T_\gamma^\# e^{i\lambda T_\gamma}}{T_\gamma|\det(I-\mathcal{P}_\gamma)|}\right). \tag{1.3}$$

The μ's appearing on the left hand side of (1.2) are the zeros of ζ_1—see [18, § 5] or [36] for an indication of this simple fact. Under certain analyticity assumptions on X and φ_t, Rugh [31] and Fried [12] showed that the Ruelle zeta function $\zeta_R(\lambda)$ is a meromorphic function of finite order but neither [13] nor our paper suggest the validity of such a statement in general.

One reason to be interested in (1.2) in the general case is the following consequence based on [19, § 4]: the counting function for the Pollicott-Ruelle resonances in wide strips cannot be sublinear. More precisely, there exists a constant C_0 such that for each $\varepsilon\in(0,1)$,

$$\#\{\mu\in\operatorname{Res}(P)\ :\ \operatorname{Im}\mu > -C_0/\varepsilon,\ |\mu|\le r\} \not\lesssim r^{1-\varepsilon}, \quad r\ge C(\varepsilon), \tag{1.4}$$

see [23] and comments below.

We arrived at the proof of main Theorem while attempting to demonstrate (1.2) for C^∞ Anosov flows. We now indicate the idea of that proof in the case of analytic continuation of $\zeta_1(\lambda)$ given by (1.3). It converges for $\operatorname{Im}\lambda \gg 1$—see Lemma 2.2 for convergence and (2.5) below for the connection to the Ruelle zeta function. The starting point is Guillemin's formula,

$$\operatorname{tr}^\flat e^{-itP} = \sum_\gamma \frac{T_\gamma^\# \delta(t-T_\gamma)}{|\det(I-\mathscr{P}_\gamma)|}, \quad t>0 \tag{1.5}$$

where the trace is defined using distributional operations of pullback by $\iota(t,x)=(t,x,x)$ and pushforward by $\pi:(t,x)\to t$: $\operatorname{tr}^\flat e^{-itP} := \pi_*\iota^* K_{e^{-itP}}$, where $K_\bullet$ denotes the distributional kernel of an operator. The pullback is well-defined in the sense of distributions [21, § 8.2] because the wave front set of $K_{e^{-itP}}$ satisfies

$$\mathrm{WF}(K_{e^{-itP}}) \cap N^*(\mathbb{R}_t \times \Delta(X)) = \varnothing, \quad t>0, \tag{1.6}$$

where $\Delta(X) \subset X \times X$ is the diagonal and $N^*(\mathbb{R}_t \times \Delta(X)) \subset T^*(\mathbb{R}_t \times X \times X)$ is the conormal bundle. See Appendix B and [17, § II] for details.

Since

$$\frac{d}{d\lambda}\log\zeta_1(\lambda) = \frac{1}{i}\sum_\gamma \frac{T_\gamma^\# e^{i\lambda T_\gamma}}{|\det(I-\mathscr{P}_\gamma)|} = \frac{1}{i}\int_0^\infty e^{it\lambda}\operatorname{tr}^\flat e^{-itP}dt,$$

it is enough to show that the right hand side has a meromorphic continuation to $\mathbb{C}$ with simple poles and residues which are non-negative integers. For that it is enough to take $t_0>0$ smaller than T_γ for all γ (note that $\operatorname{tr}^\flat e^{-itP}=0$ on $(0,t_0)$) and consider a continuation of

$$\frac{1}{i}\int_{t_0}^\infty e^{it\lambda}\operatorname{tr}^\flat e^{-itP}dt = \frac{1}{i}e^{it_0\lambda}\int_0^\infty e^{it\lambda}\operatorname{tr}^\flat \varphi_{-t_0}^* e^{-itP}dt.$$

We now note that

$$i\int_0^\infty e^{it\lambda}\varphi_{-t_0}^* e^{-itP}dt = \varphi_{-t_0}^*(P-\lambda)^{-1} \quad \text{for } \operatorname{Im}\lambda \gg 1. \tag{1.7}$$

With a justification provided by a simple approximation argument (see the proof of [22, Theorem 19.4.1] for a similar construction) it is then sufficient to continue

$$\operatorname{tr}^\flat\left(\varphi_{-t_0}^*(P-\lambda)^{-1}\right), \quad \operatorname{Im}\lambda \gg 1, \tag{1.8}$$

meromorphically. As recalled in § 3.2, $(P-\lambda)^{-1}: C^\infty(X) \to \mathscr{D}'(X)$ continues meromorphically so to check the meromorphy of (1.8) we only need to check the analogue of the wave front set relation (1.6) for the distributional kernel of $\varphi_{-t_0}^*(P-\lambda)^{-1}$, namely that this wave front set does not intersect $N^*(\Delta(X))$. But that follows from an adaptation of propagation results of Duistermaat-Hörmander [22, § 26.1], Melrose [27], and Vasy [35]. The Faure-Sjöstrand spaces [9] provide the a priori regularity which allows an application of these techniques. In fact, we use somewhat simpler anisotropic Sobolev spaces in our argument and provide an alternative approach to the meromorphic continuation of the resolvent—see §§ 3.1, 3.2.