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# LOCAL $Tb$ THEOREM WITH $L^2$ TESTING CONDITIONS AND GENERAL MEASURES: CALDERÓN-ZYGMUND OPERATORS

BY MICHAEL T. LACEY AND HENRI MARTIKAINEN

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**ABSTRACT.** – Local  $Tb$  theorems with  $L^p$  type testing conditions have been studied widely in the case of the Lebesgue measure. Such conditions are tied to the scale of the given test function's supporting cube. Until very recently, local  $Tb$  theorems in the non-homogeneous case had only been proved assuming scale invariant ( $L^\infty$  or BMO) testing conditions. Moving past such strong assumptions in non-homogeneous analysis is a key problem. In a previous paper we overcame this obstacle in the model case of square functions defined using general measures. In this paper we finally tackle the very demanding case of Calderón-Zygmund operators. That is, we prove a non-homogeneous local  $Tb$  theorem with  $L^2$  type testing conditions for all Calderón-Zygmund operators. In doing so we prove general twisted martingale transform inequalities which turn out to be subtle in our general framework.

**RÉSUMÉ.** – Les théorèmes  $Tb$  avec conditions de type  $L^p$  pour une famille de fonctions de test indexées par les cubes ont été étudiés abondamment dans le cadre de la mesure de Lebesgue. Jusqu'à très récemment, les théorèmes  $Tb$  locaux dans les espaces non doublants ont été obtenus sous des conditions invariantes par transformation affine ( $L^\infty$  ou BMO). Se dispenser de cette invariance complique la tâche. Dans un article précédent, nous avons développé une méthode permettant de surmonter cette difficulté dans un cas modèle de fonctions carrées définies à l'aide de mesures générales. Dans cet article, on s'attaque au cas des opérateurs de Calderón-Zygmund. Plus précisément, on démontre un théorème  $Tb$  local dans le cas non doublant avec des conditions de test  $L^2$  pour tous les opérateurs de Calderón-Zygmund. Un ingrédient essentiel est le contrôle d'une transformation de martingale tordue qui s'avère subtile dans notre cadre.

## 1. Introduction

In this paper we prove the boundedness of a Calderón-Zygmund operator  $T$  on  $L^2(\mu)$ , where  $\mu$  can be non-homogeneous, assuming only the existence of certain non-degenerate test functions satisfying local  $L^2$  conditions. For a given test function  $b_Q$ , associated with

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a cube  $Q \subset \mathbb{R}^n$ , the assumptions concern only the scale of  $Q$  (unlike, say, with  $L^\infty$  or BMO conditions). This is a key difficulty made much harder by the fact that we allow general measures. Indeed, such local  $Tb$  theorems with  $L^p$  testing functions are known in the homogeneous case, but proving such a result in the non-homogeneous setting is delicate. Here we are able to do this for the first time. The proof requires extensive development and usage of the techniques of non-homogeneous and two-weight dyadic analysis.

Let us begin by introducing the setting and formulate our main theorem. We assume that  $\mu$  is a measure on  $\mathbb{R}^n$  satisfying only the size condition  $\mu(B(x, r)) \lesssim r^m$  for some  $m$ . We consider Calderón-Zygmund operators  $T$  in this setting. First of all, this means that there is a kernel  $K: \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\} \rightarrow \mathbb{C}$  for which there holds for some  $C < \infty$  and  $\alpha > 0$  that

$$|K(x, y)| \leq \frac{C}{|x - y|^m}, \quad x \neq y,$$

$$|K(x, y) - K(x', y)| \leq C \frac{|x - x'|^\alpha}{|x - y|^{m+\alpha}}, \quad |x - y| \geq 2|x - x'|,$$

and

$$|K(x, y) - K(x, y')| \leq C \frac{|y - y'|^\alpha}{|x - y|^{m+\alpha}}, \quad |x - y| \geq 2|y - y'|.$$

Secondly, we demand that  $T$  is a linear operator satisfying the identity

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y) d\mu(y), \quad x \notin \text{spt } f.$$

In this paper we assume a priori that  $T: L^2(\mu) \rightarrow L^2(\mu)$  boundedly. We are after a new quantitative bound for  $\|T\|$ , independent of the a priori bound. Such practice is standard, and one can deduce to this situation by, for example, considering suitably truncated operators.

We are ready to state our main theorem—a non-homogeneous local  $Tb$  theorem with  $L^2$  type testing conditions for all Calderón-Zygmund operators.

**1.1. THEOREM.** — *Suppose that  $T: L^2(\mu) \rightarrow L^2(\mu)$  is a bounded Calderón-Zygmund operator with an adjoint operator  $T^*$ . We assume that to every cube  $Q \subset \mathbb{R}^n$  there is associated two functions  $b_Q^T$  and  $b_Q^{T^*}$  satisfying that*

1.  $\text{spt } b_Q^T \subset Q$  and  $\text{spt } b_Q^{T^*} \subset Q$ ;
2.  $\left| \int_Q b_Q^T d\mu \right| \gtrsim \mu(Q)$  and  $\left| \int_Q b_Q^{T^*} d\mu \right| \gtrsim \mu(Q)$ ;
3.  $\|b_Q^T\|_{L^2(\mu)}^2 \lesssim \mu(Q)$  and  $\|b_Q^{T^*}\|_{L^2(\mu)}^2 \lesssim \mu(Q)$ ;
4.  $\|1_Q T b_Q^T\|_{L^2(\mu)}^2 \lesssim \mu(Q)$  and  $\|1_Q T^* b_Q^{T^*}\|_{L^2(\mu)}^2 \lesssim \mu(Q)$ .

*Then we have that  $\|T\| \lesssim 1$ .*

Recently in [11] we proved a version of this theorem for square functions defined in the upper half-space. While of independent interest because of the genuinely different context, it is a result with a much simpler proof than the current one. Indeed, the square functions essentially provide a model framework where many technicalities of the Calderón-Zygmund world do not arise. One of them is that the diagonal is completely trivial for square functions while extremely delicate for Calderón-Zygmund operators. Another difference is that the recent Whitney averaging identity over good cubes of Martikainen and Mourougolou [15]

makes certain probabilistic arguments easy even in the local  $Tb$  situation. A critical difference is the fact that the paraproduct operator is much simpler in the square function case.

Before going more to the history and context, we want to discuss the proof of our main theorem, Theorem 1.1, and the references most related to our techniques. The proof is quite simply begun by reducing to the non-homogeneous  $T1$  theorem of Nazarov-Treil-Volberg [17]. More specifically, a local formulation following directly from this is used:

$$\|T\| \leq C_1 + C_2 \sup_{\substack{Q_0 \subset \mathbb{R}^n \\ Q_0 \text{ cube}}} \sup_{\substack{f, g \\ |f|, |g| \leq 1_{Q_0}}} \frac{|\langle Tf, g \rangle|}{\mu(\lambda Q_0)}.$$

Here  $\lambda > 1$  is some fixed large constant. This reduces things to proving that

$$|\langle Tf, g \rangle| \leq (C_3 + c\|T\|)\mu(\lambda Q_0),$$

where  $c$  can be taken to be arbitrarily small. Two independent random cubes  $Q^*$  and  $R^*$  for which  $Q_0 \subset Q^* \subset \lambda Q_0$  and  $Q_0 \subset R^* \subset \lambda Q_0$  are then used to expand the fixed bounded functions  $f$  and  $g$  dyadically in to martingale differences adapted to the local test functions.

We now come to the essentials. To handle the complicated paraproducts we require a non-homogeneous version of the twisted martingale difference inequalities of Auscher-Routin [2] or Lacey-Vähäkangas [13]. This is Proposition 2.4 of our current paper—a result of independent interest. Indeed, the proof of Proposition 2.4 turns out to be a demanding task. The key reason lies in the fact that even if we have performed a stopping time argument which gives us that a fixed test function  $b_F^T$  behaves nicely on a cube  $Q$  i.e.,  $\int_Q |b_F^T|^2 d\mu \lesssim \mu(Q)$ , we cannot say much what happens in the stopping children of  $Q$ . That is, in a stopping child  $Q'$  of  $Q$  we cannot use the simple argument

$$\int_{Q'} |b_F^T|^2 d\mu \leq \int_Q |b_F^T|^2 d\mu \lesssim \mu(Q) \lesssim \mu(Q')$$

which would only be available if  $\mu$  would be doubling.

Instead, the proof of Proposition 2.4 becomes about controlling *maximal truncations* of certain *half-twisted* martingales  $\sum_Q \epsilon_Q D_Q$ . Even if we are interested in an  $L^2$  result, we find it convenient to prove a weak type bound for every  $p \in (1, \infty)$  and interpolate this (the half-twisted martingales will be  $L^p$  bounded for every  $p$  unlike the original twisted martingales). But such a weak type bound can be reduced to a testing condition—an idea originally by Sawyer [18], but which can essentially also be found from e.g., [9] by Hytönen et al. The verification of this testing inequality is based crucially on controlling  $\sum_Q \epsilon_Q D_Q 1$  in  $L^p$ . This control is proved by reducing to the case  $p = 1$  using a non-homogeneous John-Nirenberg principle formulated at least by Lacey-Petermichl-Reguera [12] and Hytönen-Pérez-Treil-Volberg [8].

Proposition 2.4 is formulated in such a way that essentially the stopping generation is fixed. For this reason we perform an argument which gives that in the expansion of the pairing  $\langle Tf, g \rangle$  we can use only finitely many generations of stopping cubes. This follows from the Carleson property of the stopping cubes by noticing that the large generations provide only an absorbable error. The fact that the functions  $f$  and  $g$  are bounded plays a role in this reduction, and also later in the proof when we prove the boundedness of a certain paraproduct.