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COMPLETE PERIODICITY OF PRYM EIGENFORMS

BY ERWAN LANNEAU AND DUC-MANH NGUYEN

ABSTRACT. – This paper deals with Prym eigenforms which are introduced previously by McMullen.We prove several results on the directional flow on those surfaces, related to complete periodicity (introduced by Calta). More precisely we show that any homological direction is algebraically periodic, and any direction of a regular closed geodesic is a completely periodic direction. As a consequence we draw that the limit set of the Veech group of every Prym eigenform in some Prym loci of genus $3, 4$, and 5 is either empty, one point, or the full circle at infinity. We also construct new examples of translation surfaces satisfying the topological dichotomy (without being lattice surfaces). As a corollary we obtain new translation surfaces whose Veech group is infinitely generated and of the first kind.

RÉSUMÉ. – Dans cet article nous démontrons plusieurs résultats topologiques sur les formes propres des lieux Prym, formes différentielles abéliennes découvertes par McMullen dans des travaux antérieurs. Nous obtenons une propriété dite de complète périodicité (introduite par Calta), ainsi que de nouvelles familles de surfaces de translation vérifiant la dichotomie topologique de Veech (sans être une surface de Veech) . Comme conséquences nous montrons que l'ensemble limite des groupes de Veech de formes propres de certaines strates en genre $3, 4$, et 5 est soit vide, soit un point, soit tout le cercle à l'infini. Ceci nous permet de plus de construire de nouveaux exemples de surfaces de translation ayant un groupe de Veech infiniment engendré et de première espèce.

Notre preuve repose sur une nouvelle approche de la notion de feuilletage périodique par les involutions linéaires.

1. Introduction

1.1. Periodicity and Algebraic Periodicity

In his 1989 seminal work [35], Veech introduced an important class of translation surfaces (now called *Veech surfaces*) providing first instances of translation surfaces whose directional flows satisfy a remarkable property: for a given direction, the flow is either uniquely ergodic (all the flow lines are dense and uniformly distributed) or completely periodic (all the flow lines are closed or a saddle connection). This property is subsequently called the *Veech* *dichotomy*. Since then numerous efforts have been made in the study of the linear flows on translation surfaces, to name a very few: [22, 30, 27, 31, 11, 7]. Veech's theorem raised the issue of what can be said about the dynamics of the directional flows on non Veech surfaces.

This paper deals with the question of completely periodic linear flows. This aspect has been initiated in [5], and then developed later in [6]. A useful invariant to detect completely periodic flows (i.e., all the flow lines are closed or connect singularities), introduced in Arnoux's thesis [1], is the Sah-Arnoux-Fathi (SAF) invariant. It is well known that the linear flow $\mathcal{F}_\theta$ in a direction $\theta \in \mathbb{RP}^1$ on a translation surface (X,ω) (equipped with a transversal interval I) provides an interval exchange transformation T_θ, which is the first return map to I. The invariant of the flow in direction θ can be informally defined by

$$SAF(T_\theta) = \int_I 1 \otimes (T_\theta(x) - x)dx \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$$

(the integral is actually a finite sum). If $\mathcal{F}_\theta$ is periodic, that is when every leaf of $\mathcal{F}_\theta$ is either a closed curve or an interval joining two zeros of ω, then $SAF(T_\theta) = 0$. However the converse is not true in general. Following this remark, the direction θ will be called *algebraically periodic* if the SAF-invariant of the flow $\mathcal{F}_\theta$ vanishes.

A translation surface (X,ω) is *completely periodic (in the sense of Calta)* if for every $\theta \in \mathbb{RP}^1$ for which $\mathcal{F}_\theta$ has a closed regular orbit, the flow $\mathcal{F}_\theta$ is completely periodic. We have the corresponding "algebraic" definition: the surface (X,ω) is *completely algebraically periodic* if the SAF-invariant of $\mathcal{F}_\theta$ vanishes in any homological direction ($\theta \in \mathbb{RP}^1$ is *homological* if it is the direction of a vector $\int_c \omega \in \mathbb{C} \simeq \mathbb{R}^2$ for some $c \in H_1(X,\mathbb{Z})$). These notions are introduced in [5] and [6].

Flat tori and their ramified coverings are both completely periodic and completely algebraically periodic; in this case, up to a renormalization by $\mathrm{GL}^+(2,\mathbb{R})$, the set of homological directions is $\mathbb{Q} \cup \{\infty\}$. In [5], Calta proved that these two properties also coincide for genus 2 translation surfaces, in which case the set of homological directions is $K\mathbb{P}^1$, where K is either $\mathbb{Q}$ or a real quadratic field over $\mathbb{Q}$, and moreover a surface in $\mathcal{H}(2)$ is completely periodic if and only if it is a Veech surface (see also [25]). However there are completely periodic surfaces in $\mathcal{H}(1,1)$ that are not Veech surfaces (actually, most of them are not Veech surfaces).

We will say that a quadratic differential is algebraically completely periodic (respectively, completely periodic in the sense of Calta) if its orientation double cover is. Translation surfaces in genus two are closely related to quadratic differentials over $\mathbb{CP}^1$, since we have the following identifications (which are $\mathrm{GL}^+(2,\mathbb{R})$ invariant) $\mathcal{H}(2) \simeq \mathcal{Q}(-1^5,1)$, $\mathcal{H}(1,1) \simeq \mathcal{Q}(-1^6,2)$. Note that $\dim_{\mathbb{C}} \mathcal{Q}(-1^5,1) = 4$, and $\dim_{\mathbb{C}} \mathcal{Q}(-1^6,2) = 5$. We record all strata of quadratic differentials of dimension 5 in Table 1. In this paper, our first aim is to extend Calta's result to all of these strata.

THEOREM A. – *Let (Y,q) be quadratic differential in one of the strata in Table 1. If (Y,q) is completely algebraically periodic then it is completely periodic in the sense of Calta.*

THEOREM B. – *Let K be either $\mathbb{Q}$ or a real quadratic field. For any stratum $\mathcal{Q}(\kappa)$ in Table 1, the set of algebraically completely periodic quadratic differentials in $\mathcal{Q}(\kappa)$, with homological directions in $K\mathbb{P}^1$ up to renormalization by* $\mathrm{GL}^+(2,\mathbb{R})$ *is a union of* $\mathrm{GL}^+(2,\mathbb{R})$*-invariant* *submanifolds of complex dimension* 3*. Such invariant submanifolds are called* Prym eigenform loci *(see Section 2 for precise definitions).*

The techniques developed in this paper for the proof of Theorems A and B actually provide us with some precise information on the flow in directions for which the SAF-invariant vanishes: we get some topological properties of the directional flows on surfaces in some particular strata. Here we introduce the terminology of [7].

We say that a translation surface satisfies the *topological dichotomy* if for every direction, either the flow is minimal, or every flow line is closed or a saddle connection. Observe that this is equivalent to saying that if there is a saddle connection in some direction, then there is a cylinder decomposition of the surface in that direction. Obviously a Veech surface satisfies the topological dichotomy. First examples of surfaces satisfying topological dichotomy without being Veech surfaces have been constructed in [12] (see also [23]). All examples are ramified coverings above "true" Veech surfaces. Our next theorem provides us with new examples which do not arise from a covering construction above Veech surfaces (see Theorem 1.13).

THEOREM C. – *Let (Y,q) be a quadratic differential in $\mathcal{Q}(8)$ or $\mathcal{Q}(-1,2,3)$. Assume that all the periods (relative and absolute) of the orientation double cover of (Y,q) belong to $K(\imath)$, where K is either $\mathbb{Q}$ or a real quadratic field. If (Y,q) is algebraically completely periodic then it satisfies the topological dichotomy. In particular, if (Y,q) is stabilized by a pseudo-Anosov homeomorphism, then it satisfies the topological dichotomy.*

Observe that Theorem C is false for other strata. Moreover, "most" of surfaces of Theorem C are not Veech surfaces, namely:

THEOREM D. – *The following two hold:*

(1) *There are quadratic differentials in the strata $\mathcal{Q}(8)$ and $\mathcal{Q}(-1,2,3)$ satisfying the topological dichotomy without being Veech surfaces.*
(2) *There are quadratic differentials in each of the strata $\mathcal{Q}(-1^3,1,2)$, $\mathcal{Q}(-1,2,3)$, and $\mathcal{Q}(8)$ whose Veech group is infinitely generated and of the first kind.*

Finally our techniques also provide us with the following result for quadratic differentials in a slightly larger family of strata (compared to Theorem C).

THEOREM E. – *For any quadratic differential in the collection of strata $\mathcal{Q}(-1^3,1,2)$, $\mathcal{Q}(-1,2,3)$, and $\mathcal{Q}(8)$ the limit set of its Veech group is either the empty set, a single point, or the full circle at infinity.*

1.2. Prym loci and Prym eigenforms

From the work of McMullen [25], it turns out that all completely periodic surfaces in genus two belong to the loci of *eigenforms for real multiplication*. Later McMullen [24] proved the existence of similar loci in genus $3, 4$ and 5. These loci are of interest since they are closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant sub-manifolds in the moduli spaces of Abelian differentials. We briefly recall the definitions of those objects here below.