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QUADRATIC CAPELLI OPERATORS AND OKOUNKOV POLYNOMIALS

BY Siddhartha SAHI AND Hadi SALMASIAN

ABSTRACT. – Let Z be the symmetric cone of $r \times r$ positive definite Hermitian matrices over a real division algebra $\mathbb{F}$. Then Z admits a natural family of invariant differential operators—the *Capelli operators* C_λ—indexed by partitions λ of length at most r, whose eigenvalues are specializations of Knop-Sahi interpolation polynomials.

In this paper we consider a double fibration $Y \longleftarrow X \longrightarrow Z$ where Y is the Grassmanian of r-dimensional subspaces of $\mathbb{F}^n$ with $n \geq 2r$. Using this we construct a family of invariant differential operators $D_{\lambda,s}$ on Y that we refer to as *quadratic* Capelli operators. Our main result shows that the eigenvalues of the $D_{\lambda,s}$ are specializations of Okounkov interpolation polynomials.

RÉSUMÉ. – Soit Z le cône symétrique de matrices de tailles $r \times r$ hermitiennes positives sur une algèbre de division réelle $\mathbb{F}$. Alors Z admet une famille naturelle d'opérateurs différentiels invariants — les *Opérateurs de Capelli* C_λ — indexés par des partitions λ de longueur au plus r, dont les valeurs propres sont des spécialisations de polynômes d'interpolation Knop-Sahi.

Dans cet article, nous considérons une double fibration $Y \longleftarrow X \longrightarrow Z$ où Y est la variété grassmanienne des sous-espaces de dimension r de $\mathbb{F}^n$ avec $n \geq 2r$. En utilisant cela, nous construisons une famille d'opérateurs différentiels invariants $D_{\lambda,s}$ sur Y que nous appelons opérateurs de Capelli *quadratiques*. Notre résultat principal montre que les valeurs propres des $D_{\lambda,s}$ sont des spécialisations de polynômes d'interpolation Okounkov.

1. Introduction

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be a real division algebra. Fix integers r and n such that $1 \leq r \leq \frac{n}{2}$. Let Y be the Grassmannian of r-dimensional subspaces of $\mathbb{F}^n$, and let Z be the symmetric cone of $r \times r$ positive definite Hermitian $\mathbb{F}$-matrices. Then one has a double fibration

$$\begin{array}{ccc} & X & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ Y & & Z, \end{array}$$

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where X is the space of $n \times r$ matrices of $\mathbb{F}$-rank r. For $x \in X$, $\varphi(x)$ is the column space (or range) of x, while $\psi(x) := x^\dagger x$, where $x^\dagger$ denotes the $\mathbb{F}$-Hermitian adjoint of x.

One can give another description of the above structure in terms of the groups

$$G_m := \mathrm{GL}_m(\mathbb{F}), \quad K_m := \mathrm{U}_m(\mathbb{F}) := \left\{g \in G_m \ : \ g^\dagger g = I_{m\times m}\right\}.$$

The groups K_n and G_r act on X by matrix multiplication on the left and right respectively, and the maps $X \xrightarrow{\psi} Z$ and $X \xrightarrow{\varphi} Y$ are simply the corresponding quotient maps. Moreover, $X \xrightarrow{\varphi} Y$ is a principal G_r-bundle, while $X \xrightarrow{\psi} Z$ is a fibration whose fibers are isomorphic to the Stiefel manifold K_n/K_{n-r}. Also, since the actions of K_n and G_r on X commute, it follows that G_r acts on Z, and K_n acts on Y. In fact, Y and Z are symmetric spaces for the latter actions. More precisely, we have

$$Y \simeq K_n/(K_r \times K_{n-r}) \text{ and } Z \simeq G_r/K_r.$$

The cone Z is a symmetric space of type A, and admits an important basis of G_r-invariant differential operators C_λ, indexed by partitions $\lambda \in \mathscr{P}_r$, where

$$\mathscr{P}_r := \{(\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r \ : \ \lambda_1 \geq \dots \geq \lambda_r \geq 0\}.$$

The operators C_λ were first studied by the first author in [20], and were referred to as *Capelli operators*. It is known that the spectrum of C_λ is given by specialization of Knop-Sahi type A interpolation polynomials [20], [12], [30].

On the other hand, Y is a compact symmetric space of type BC. In this paper, we use the above double fibration to construct a family of K_n-invariant differential operators $D_{\lambda,s}$ on Y that correspond to the Capelli operators C_λ. We call the operators $D_{\lambda,s}$ the *quadratic Capelli operators* because they are obtained from C_λ by pullback of the quadratic map ψ. Our main result proves that the spectrum of $D_{\lambda,s}$ is given by specialization of the *Okounkov* type BC interpolation polynomials $P_\lambda(x;\tau,\alpha)$ (see [14, Sec. 5.3] and [19]).

To describe our main result precisely, we begin by introducing some notation. Set $K := K_n$ and $M := K_r \times K_{n-r} \subset K$, so that $Y \simeq K/M$. The group K acts by left translation on $C^\infty(Y)$, the space of complex-valued smooth functions on Y. The operators $D_{\lambda,s}$ leave the subspace $C^\infty(Y)_{K\text{-finite}}$ of K-finite vectors invariant. By standard results from the theory of compact symmetric spaces (for example, see [5, Chap. V]), $C^\infty(Y)_{K\text{-finite}}$ decomposes as a multiplicity-free direct sum of irreducible M-spherical K-modules, which are naturally parametrized by partitions $\mu \in \mathscr{P}_r$. Our next goal is to describe this parametrization. Let $\mathfrak{k}$ and $\mathfrak{m}$ denote the Lie algebras of K and M. Fix a Cartan decomposition $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a Cartan subspace, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$ such that $\mathfrak{a} \subseteq \mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{t} := \mathfrak{h} \cap \mathfrak{m}$. Set $\mathfrak{k}_\mathbb{C} := \mathfrak{k} \otimes_\mathbb{R} \mathbb{C}$, $\mathfrak{h}_\mathbb{C} := \mathfrak{h} \otimes_\mathbb{R} \mathbb{C}$, $\mathfrak{a}_\mathbb{C} := \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$, and $\mathfrak{t}_\mathbb{C} := \mathfrak{t} \otimes_\mathbb{R} \mathbb{C}$. The restricted root system $\Sigma := \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{a}_\mathbb{C})$ is of type BC_r. We choose a positive system $\Sigma^+ \subset \Sigma$ and a basis $\mathsf{e}_1, \dots, \mathsf{e}_r$ of $\mathfrak{a}_\mathbb{C}^*$ such that the multiplicity m_α of every $\alpha \in \Sigma^+$ is given in terms of n, r, and $d := \dim \mathbb{F}$ in Table 1 below.

α	$\mathsf{e}_i,\ 1 \leq i \leq r$	$\mathsf{e}_i \pm \mathsf{e}_j,\ 1 \leq i < j \leq r$	$2\mathsf{e}_i,\ 1 \leq i \leq r$
m_α	$d(n-2r)$	d	$d-1.$

Table 1.

We also choose a positive system for the root system $\Delta := \Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which is compatible with Σ^+. Let $\widetilde{\mu} \in \mathfrak{h}_{\mathbb{C}}^*$. By the Cartan-Helgason theorem, $\widetilde{\mu}$ is the highest weight of an irreducible M-spherical K-module if and only if

$$\widetilde{\mu}|_{\mathfrak{t}_{\mathbb{C}}} = 0 \text{ and } \widetilde{\mu}|_{\mathfrak{a}_{\mathbb{C}}} = \sum_{i=1}^{r} 2\mu_i \mathsf{e}_i, \text{ where } \mu := (\mu_1, \dots, \mu_r) \in \mathscr{P}_r. \tag{1}$$

REMARK 1.1. – Assume that $\mathbb{F} = \mathbb{R}$. Then K is disconnected, and if K° denotes the connected component of identity of K, then $M \cap K^\circ$ is also disconnected. Therefore the Cartan-Helgason theorem as stated for instance in [5, Cor. V.4.2] does not apply immediately to the case $\mathbb{F} = \mathbb{R}$. However, one can use the refinement of the Cartan-Helgason theorem for the pair $(K^\circ, M \cap K^\circ)$, given in [4, Sec. 12.3.2], as well as the description of irreducible representations of K in terms of irreducible representations of K°, given in [4, Thm 5.5.23], to obtain the condition (1).

From now on, we denote the M-spherical K-module with highest weight $\widetilde{\mu}$ satisfying (1) by V_μ. Therefore as K-modules,

$$C^\infty(Y)_{K\text{-finite}} \simeq \bigoplus_{\mu \in \mathscr{P}_r} V_\mu.$$

The operator $D_{\lambda,s}$ acts on V_μ by the scalar

$$c_{\lambda,s}(\mu) := \mathrm{HC}(D_{\lambda,s}) \left(\widetilde{\mu}|_{\mathfrak{a}_{\mathbb{C}}} + \rho \right),$$

where $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$, $\widetilde{\mu}$ is the highest weight of V_μ, and $\mathrm{HC} : \mathbf{D}_K(Y) \to \mathscr{P}(\mathfrak{a}_{\mathbb{C}}^*)^{\mathsf{W}}$ is the Harish-Chandra homomorphism from the algebra $\mathbf{D}_K(Y)$ of K-invariant differential operators on Y onto the algebra of polynomials on $\mathfrak{a}_{\mathbb{C}}^*$ that are invariant under the action of the restricted Weyl group W.

We now recall the definition of the Okounkov polynomials $P_\lambda(x; \tau, \alpha)$. Let $\mathbb{k} := \mathbb{C}(\tau, \alpha)$ denote the field of rational functions in τ and α. Let $\delta, \mathbb{1} \in \mathscr{P}_r$ and $\varrho_{\tau,\alpha} \in \mathbb{k}^r$ be defined by

$$\delta := (r-1, \dots, 0), \quad \mathbb{1} := (1, \dots, 1), \quad \varrho_{\tau,\alpha} := \tau\delta + \alpha\mathbb{1}. \tag{2}$$

For $\lambda \in \mathscr{P}_r$, we define $|\lambda| := \sum_{i=1}^{r} \lambda_i$. Up to a scalar, $P_\lambda(x; \tau, \alpha) \in \mathbb{k}[x_1, \dots, x_r]$ is the unique polynomial of degree $2|\lambda|$, which is invariant under permutations and sign changes of $x_1, \dots, x_r$, and satisfies

$$P_\lambda(\mu + \varrho_{\tau,\alpha}; \tau, \alpha) = 0$$

for every $\mu \in \mathscr{P}_r$ such that $|\mu| \le |\lambda|$ and $\mu \ne \lambda$ (for more details, see Section 4).

Recall that $d := \dim \mathbb{F}$. Let $\mathbb{i} : \mathbb{C}^r \to \mathfrak{a}_{\mathbb{C}}^*$ be the linear map defined by $\mathbb{i}(\mathsf{e}^i) := 2\mathsf{e}_i$ for $1 \le i \le r$, where $\mathsf{e}^1, \dots, \mathsf{e}^r$ are the standard basis vectors of $\mathbb{C}^r$ (therefore $\mathbb{i}(\mathscr{P}_r)$ is the set of restrictions to $\mathfrak{a}_{\mathbb{C}}$ of highest weights of M-spherical K-modules). Set

$$\varrho := \mathbb{i}^{-1}(\rho).$$

A simple calculation yields

$$\varrho = (\varrho_1, \dots, \varrho_r) \text{ where } \varrho_i := \frac{dn}{4} - \frac{1}{2} - \frac{d(i-1)}{2} \text{ for every } 1 \le i \le r. \tag{3}$$

We are now ready to state our main theorem.