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RINGS OF SEPARATED POWER SERIES AND QUASI-AFFINOID GEOMETRY

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Leonard Lipshitz, Zachary Robinson

Abstract. — The papers in this volume present a theory of rigid analytic geometry over an ultrametric field K that generalizes the classical, affinoid, theory to the setting of relative rigid analytic geometry over an "open" polydisc. The theory is based on the commutative algebra of power series rings $S_{m,n}$ that is developed in the first paper in this volume, *Rings of Separated Power Series*. Quasi–affinoid algebras (quotients $S_{m,n}/I$) share many properties with affinoid algebras (quotients T_m/I of a ring of strictly convergent power series.) Among the principal results are the Nullstellensatz for quasi–affinoid algebras A and the Universal Property for a broad class of open subdomains of Max A, the R–subdomains. The second paper, *Model Completeness and Subanalytic Sets*, obtains a structure theory for images of analytic maps based on any subcollection of $S = \cup S_{m,n}$ that satisfies certain closure properties; for example $T = \cup T_m$. The argument exploits the existential definability of the Weierstrass data as well as a difference between affinoid and quasi–affinoid rigid analytic geometry; namely, that a quasi–affinoid variety Max A in general may be covered by finitely many disjoint quasi–affinoid subdomains, just as the valuation ring K° is the union of its maximal ideal $K^{\circ\circ}$ and its multiplicative units. A crucial role is played by the theory of generalized rings of fractions developed in the first paper. The third paper, *Quasi–Affinoid Varieties*, defines the category of $S_{m,n}$–analytic varieties $X = \text{Max } A$ and establishes the acyclicity of quasi–affinoid covers. The proofs employ results from the first paper; in particular, the fact that the assignment $U \mapsto \mathcal{O}_X(U)$ is a presheaf of A–algebras for R–subdomains U of X. The quantifier elimination of the second paper is used to relate quasi–affinoid and affinoid covers, a key step in the proof of the Acyclicity Theorem. The fourth paper, *A Rigid Analytic Approximation Theorem*, gives a global Artin Approximation theorem between a "Henselization" $H_{m,n}$ of a ring T_{m+n} of strictly convergent power series and its "completion" $S_{m,n}$. This links the algebraic properties of affinoid and quasi–affinoid algebras.

***Résumé* (Anneaux de séries séparées et géométrie quasi-affinode)**

Les articles de ce volume présentent une théorie de la géométrie analytique rigide sur un corps ultramétrique K qui généralise la théorie affinoïde classique au cas de la géométrie analytique rigide relative sur un polydisque « ouvert ». Cette théorie est basée sur l'étude algébrique des anneaux de séries convergentes $S_{m,n}$ développée dans le premier article, *Rings of Separated Power Series*. Les algèbres quasi-affinoïdes (les quotients $S_{m,n}/I$) partagent de nombreuses propriétés avec les algèbres affinoïdes (les quotients T_m/I d'un anneau de séries strictement convergentes). Parmi les résultats principaux signalons le Nullstellensatz pour les algèbres quasi-affinoïdes A ainsi que la Propriété Universelle pour une large classe de sous-domaines ouverts de Max X, les R-sous-domaines. Le second article, *Model Completeness and Subanalytic Sets*, contient des résultats sur la structure des images de familles de fonctions analytiques provenant par extension d'une famille quelconque de fonctions de $S = \cup S_{m,n}$ satisfaisant certaines propriétés de fermeture ; par exemple $T = \cup T_m$ est une telle famille. La preuve utilise le fait que les données de Weierstrass sont définissables ainsi que le fait, témoignant de la différence entre géométrie affinoïde et quasi-affinoïde, qu'une variété quasi-affinoïde Max A peut généralement être recouverte par un nombre fini de sous-domaines quasi-affinoïdes disjoints, de la même façon que l'anneau de valuation K° est l'union de son idéal maximal $K^{\circ\circ}$ et de ses unités multiplicatives. La théorie des anneaux généralisés de fractions développée dans le premier article joue un rôle crucial. Dans le troisième article, *Quasi-Affinoid Varieties*, on définit la catégorie des variétés $S_{m,n}$-analytiques $X = \operatorname{Max} A$ et on établit l'acyclicité des recouvrements quasi-affinoïdes. Les démonstrations emploient des résultats du premier article, notamment le fait que le foncteur $U \mapsto \mathcal{O}_X(U)$ est un préfaisceau d'A-algèbres pour des R-sous-domaines U de X. On utilise également le résultat d'élimination des quantificateurs obtenu dans le second article pour établir un rapport entre les recouvrements quasi-affinoïdes et les recouvrements affinoïdes, ce qui est une étape cruciale dans la démonstration du théorème d'acyclicité. Le quatrième article, *A Rigid Analytic Approximation Theorem*, donne un théorème d'approximation globale d'Artin entre un « hensélisé » $H_{m,n}$ d'un anneau T_{m+n} de séries strictement convergentes et son complété $S_{m,n}$. Ce résultat permet de relier les propriétés algébriques des algèbres quasi-affinoïdes et affinoïdes.

CONTENTS

INTRODUCTION

Let K be a field, complete with respect to the non-trivial ultrametric absolute value $|\cdot| : K \to \mathbb{R}_+$. By K° denote the valuation ring, by $K^{\circ\circ}$ its maximal ideal, and by $\widetilde{K}$ the residue field $K^\circ/K^{\circ\circ}$. Let K' be an algebraically closed field containing K and consider the polydisc

$$\Delta_{m,n} := ((K')^\circ)^m \times ((K')^{\circ\circ})^n .$$

In 1961, Tate introduced rings T_m of analytic functions on the closed polydiscs $\Delta_{m,0}$. These rings lift the affine algebraic geometry of the field $\widetilde{K}$. In particular, the Euclidean Division Theorem for $\widetilde{K}[\xi]$ lifts to a global Weierstrass Division Theorem for T_m. The basic properties of T_m that follow from Weierstrass Division include Noetherianness, Noether Normalization, unique factorization, and a Nullstellensatz. These results pave the way for the development of rigid (affinoid) analytic geometry.

The representation

$$\Delta_{m,n} = \varinjlim_{\varepsilon} ((K')^\circ)^m \times (\varepsilon(K')^\circ)^n,$$

where $\varepsilon \in (K')^{\circ\circ}$, yields a ring of analytic functions on $\Delta_{m,n}$ by taking a corresponding inverse limit of Tate rings. This gives the polydisc $\Delta_{m,n}$ the structure of a rigid analytic variety. But its global functions are, in general, unbounded. Even if one restricts attention to those functions with finite supremum norm, the geometric behavior can be pathological.

In the first paper, *Rings of Separated Power Series*, we define rings $S_{m,n}$ of bounded analytic functions on $\Delta_{m,n}$ with a tractable algebraic and geometric behavior. Those rings share many of the nice properties of the Tate rings T_m, though the proofs are often rather more difficult. We show that the rings $S_{m,n}$ are Noetherian rings (often K-Banach algebras) of bounded analytic functions on $\Delta_{m,n}$, that satisfy a Nullstellensatz, are unique factorization domains, and are regular rings of dimension $m + n$.

We call quotients of the $S_{m,n}$ *quasi-affinoid algebras*. Quasi-affinoid algrebras share most of the properties of affinoid algebras. For example, the residue norms arising from different presentations of a quasi-affinoid algebra are all equivalent; quasi-affinoid morphisms are continuous; in characteristic zero (and often in characteristic p) the residue norms and the supremum norm on a reduced quasi-affinoid algebra are equivalent; and quasi-affinoid rational domains satisfy the appropriate universal mapping property. These results pave the way for the development of a relative rigid analytic geometry over open polydiscs.

We give three applications of the general theory. In the second paper, *Model Completeness and Subanalytic Sets*, we present a quantifier elimination theorem which lays the foundation for the theory of rigid subanalytic sets based on the Tate Rings. The third paper, *Quasi-Affinoid Varieties*, applies the results of the first two papers to treat the basic sheaf theory of quasi-affinoid varieties and to prove the quasi-affinoid Acyclicity Theorem. In the fourth paper, *A Rigid Analytic Approximation Theorem*, a global Artin Approximation Theorem is presented for the pair of rings $H_{m,n} \hookrightarrow S_{m,n}$, where $H_{m,n}$ is the algebraic closure of T_{m+n} in $S_{m,n}$. In this context the rings $S_{m,n}$ play the role of a kind of completion of the Tate rings.

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RINGS OF SEPARATED POWER SERIES

1. Introduction

Let K be a field, complete with respect to the non-trivial ultrametric absolute value $|\cdot| : K \to \mathbb{R}_+$. By K° denote the valuation ring, by $K^{\circ\circ}$ its maximal ideal, and by $\widetilde{K}$ the residue field $K^\circ/K^{\circ\circ}$. Let K' be an algebraically closed field containing K and consider the polydisc

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In 1961, Tate [**39**] introduced rings T_m of analytic functions on the closed polydiscs $\Delta_{m,0}$. These rings lift the affine algebraic geometry of the field $\widetilde{K}$. In particular, the Euclidean Division Theorem for $\widetilde{K}[\xi]$ lifts to a global Weierstrass Division Theorem for T_m. The basic properties of T_m that follow from Weierstrass Division include Noetherianness, Noether Normalization, unique factorization, and a Nullstellensatz. These results pave the way for the development of rigid analytic geometry (see [**6**] and [**10**]).

Because in its metric topology K' is totally disconnected and not locally compact, to construct rigid analytic spaces one relies on a Grothendieck topology to provide a suitable framework for sheaf theory. For example, the basic admissible open affinoids of rigid analytic geometry are obtained by an analytic process analogous to localization in algebraic geometry (see [**6**, Section 7.2.3]). The resulting domains, rational domains, satisfy a certain universal property (see [**6**, Section 7.2.2]) and therefore give a local theory of rigid analytic spaces. The local data are linked together with a notion of admissible open cover and Tate's Acyclicity Theorem. This makes it possible, for example, to endow every algebraic variety over K with an analytic structure, that of a rigid analytic variety.

The representation

$$\Delta_{m,n} = \varinjlim_{\varepsilon} ((K')^\circ)^m \times (\varepsilon(K')^\circ)^n,$$

where $\varepsilon \in (K')^{\circ\circ}$, yields a ring of analytic functions on $\Delta_{m,n}$ by taking a corresponding inverse limit of Tate rings. This gives the polydisc $\Delta_{m,n}$ the structure of a rigid analytic variety. But its global functions are, in general, unbounded. Even if one restricts attention to those functions with finite supremum norm, the geometric behavior can be pathological. For example, let $\{a_i\}_{i\in\mathbb{N}} \subset (K')^{\circ\circ}$ be a sequence such that $\lim_{i\to\infty} |a_i| = 1$. Put

$$f(\rho) := \sum a_i \rho^i.$$

Then f converges and has infinitely many zeros on $\Delta_{0,1}$. This follows by restricting to the closed subdiscs $\varepsilon \cdot \Delta_{1,0}$ and applying Weierstrass Preparation.

The rings $S_{m,n}$, defined below, represent Noetherian rings (often, K-Banach algebras) of bounded analytic functions on $\Delta_{m,n}$ with a tractable algebraic and geometric behavior. We address the issue of the corresponding sheaf theory in [**22**].

These rings have been used in various contexts. In [**16**], where the $S_{m,n}$ were first defined, they were used to obtain a uniform bound on the number of isolated points in fibers of affinoid maps. This result was strengthened in [**2**] to give a uniform bound on the piece numbers of such fibers. In [**11**], rings $S_{0,n}$ were used to lift the rings $\widetilde{K}[\![\rho]\!]$ in order to obtain analytic information about local rings of algebraic varieties over $\widetilde{K}$. In [**17**] (and later in [**21**]), the $S_{m,n}$ were used to provide the basis for a theory of rigid subanalytic sets; i.e., images of K-analytic maps. This theory of rigid subanalytic sets was developed considerably further in [**21**], [**19**], [**18**], [**20**]. The manuscript [**21**] (unpublished) contains a quantifier simplification theorem suitable for the development of a theory of subanalytic sets based on the Tate rings. That manuscript was produced in 1995, well before the completion of this paper, and hence it was written to be self contained. As a result the proofs were rather ad hoc. In the paper [**23**] we give a smoother and more general treatment of that quantifier simplification theorem, based on some of the machinery developed in this paper, specifically the Weierstrass Division and Preparation Theorems (Theorem 2.3.8 and Corollary 2.3.9) and the concept of "generalized ring of fractions" developed in Section 5.

(The theory of the images of semianalytic sets under proper K-analytic maps was developed by Schoutens in [**32**]–[**36**]. Recently in [**12**], [**37**] and [**13**] Gardener and Schoutens have given a quantifier elimination in the language of Denef and van den Dries [**9**] over the Tate rings T_m, using the results of Raynaud–Mehlmann [**27**], Berkovich [**3**], and Hironaka, [**15**]. The proof of their elimination theorem also depends on the model completeness result of [**21**], see [**23**, Section 4].)

The theory of the rings $S_{m,n}$ was not developed systematically in papers [**16**], [**17**], [**18**], [**19**], [**20**] and [**21**]. Instead, partial results were proved as needed. The accumulation of these partial results convinced us that a systematic theory of the rings $S_{m,n}$ would be possible and would provide a natural basis for rigid analytic geometry on the polydiscs $\Delta_{m,n}$. The theory developed in this paper has been applied in [**23**]