ASTÉRISQUE 272

GEOMETRIZATION OF 3-ORBIFOLDS OF CYCLIC TYPE

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with an Appendix :

Limit of hyperbolicity for spherical 3-orbifolds

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Société Mathématique de France 2001
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification.* — 57M50, 57M60, 53C20, 53C23.

Key words and phrases. — Orbifold, hyperbolic, cone manifold, collapse, simplicial volume, Kleinian group.

Both authors have been strongly supported by the European TMR ERBFMRXCT 960040 *Singularidades de ecuaciones diferenciales y foliaciones.* The second named author was partially supported by DGICT through project PB96-1152.

GEOMETRIZATION OF 3-ORBIFOLDS OF CYCLIC TYPE

Michel Boileau and Joan Porti

with the collaboration of **Michael Heusener**

Abstract. **—** We prove the orbifold theorem in the cyclic case: If $\mathcal{O}$ is a compact oriented irreducible atoroidal 3-orbifold whose ramification locus is a non-empty submanifold, then $\mathcal{O}$ is geometric, i.e. it has a hyperbolic, a Euclidean or a Seifert fibred structure. This theorem implies Thurston's geometrization conjecture for compact orientable irreducible three-manifolds having a non-free symmetry.

Résumé **(Géométrisation des orbi-variétés tridimensionnelles de type cyclique)**

Nous démontrons le théorème des orbi-variétés de Thurston dans le cas cyclique : une orbi-variété tridimensionelle, compacte, orientable, irréductible, atoroïdale et dont le lieu de ramification est une sous-variété non vide, admet soit une structure hyperbolique ou Euclidienne, soit une fibration de Seifert. Ce théorème implique qu'une variété tridimensionelle, compacte, irréductible et possédant une symétrie non libre, vérifie la conjecture de géométrisation de Thurston.

CONTENTS

INTRODUCTION

A 3-dimensional orbifold is a metrizable space with coherent local models given by quotients of $\mathbb{R}^3$ by finite subgroups of $O(3)$. For example, the quotient of a 3-manifold by a properly discontinuous group action naturally inherits a structure of a 3-orbifold. When the group action is finite, such an orbifold is said to be *very good*. For a general background about orbifolds see [**BS1**], [**BS2**], [**DaM**], [**Kap**, Chap. 7], [**Sc3**], [**Tak1**] and [**Thu1**, Chap. 13].

The purpose of this monograph is to give a complete proof of Thurston's orbifold theorem in the case *where all local isotropy groups are cyclic subgroups of* $SO(3)$. Following [**DaM**], we say that such an orbifold is of *cyclic type* when in addition the ramification locus is non-empty. Hence a 3-orbifold $\mathcal{O}$ is of cyclic type iff its ramification locus Σ is a non-empty 1-dimensional submanifold of the underlying manifold $|\mathcal{O}|$, which is transverse to the boundary $\partial|\mathcal{O}| = |\partial\mathcal{O}|$. The first result presented here is the following version of Thurston's Orbifold Theorem:

Theorem 1. — *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible, and ∂-incompressible* 3*-orbifold of cyclic type. If $\mathcal{O}$ is very good, topologically atoroidal and acylindrical, then $\mathcal{O}$ is geometric (i.e. $\mathcal{O}$ admits either a hyperbolic, a Euclidean, or a Seifert fibred structure).*

Remark. — When $\partial\mathcal{O}$ is a union of toric 2-suborbifolds, the hypothesis that $\mathcal{O}$ is acylindrical is not needed.

If $\partial\mathcal{O} \neq \varnothing$ and $\mathcal{O}$ is not I-fibred, then $\mathcal{O}$ admits a hyperbolic structure of finite volume with totally geodesic boundary and cusps.

We only consider *smooth* orbifolds, so that the local isotropy groups are always orthogonal. We recall that an orbifold is said to be *good* if it has a covering which is a manifold. Moreover if this covering is finite then the orbifold is said to be *very good*.

A general compact orientable irreducible and atoroidal 3-orbifold (which is not a priori very good) can be canonically split along a maximal (perhaps empty) collection of disjoint and non-parallel hyperbolic turnovers (i.e. a 2-orbifold with underlying space a 2-sphere and with three branching points) into either *small* or *Haken* 3-suborbifolds.

An orientable compact 3-orbifold $\mathcal{O}$ is *small* if it is irreducible, its boundary $\partial\mathcal{O}$ is a (perhaps empty) collection of turnovers, and $\mathcal{O}$ does not contain any essential orientable 2-suborbifold.

Using Theorem 1, we are able to geometrize such small 3-orbifolds, and hence to show that they are in fact very good.

Theorem 2. — *Let $\mathcal{O}$ be a compact, orientable, connected, small 3-orbifold of cyclic type. Then $\mathcal{O}$ is geometric.*

Therefore, to get a complete picture (avoiding the very good hypothesis), it remains to geometrize the Haken atoroidal pieces.

An orientable compact 3-orbifold $\mathcal{O}$ is *Haken* if:

- $\mathcal{O}$ is irreducible,
- every embedded turnover is parallel to the boundary
- and $\mathcal{O}$ contains an embedded orientable incompressible 2-suborbifold different from a turnover.

The geometrization of Haken atoroidal 3-orbifolds relies on the following extension of Thurston's hyperbolization theorem (for Haken 3-manifolds):

***Theorem 3* (Thurston's hyperbolization theorem)**. — *Let $\mathcal{O}$ be a compact, orientable, connected, irreducible, Haken 3-orbifold. If $\mathcal{O}$ is topologically atoroidal and not Seifert fibred, nor Euclidean, then $\mathcal{O}$ is hyperbolic.*

It is a result of W. Dunbar [**Dun2**] that an orientable Haken 3-orbifold can be decomposed into either discal 3-orbifolds or thick turnovers (i.e. $\{\text{turnovers}\}\times[0,1]$) by repeated cutting along 2-sided properly embedded essential 2-suborbifolds.

Due to this fact, the proof of Theorem 3 follows exactly the scheme of the proof for Haken 3-manifolds [**Thu2, Thu3, Thu5**], [**McM1**], [**Kap**], [**Ot1, Ot2**]. We do not give a detailed proof of it here, but we only present the main steps to take in consideration and indicate shortly how to handle them in Chapter 8.

Since hyperbolic turnovers are rigid, Theorem 2 and Theorem 3 imply Thurston's orbifold theorem in the cyclic type case:

Thurston's Orbifold Theorem. — *Let $\mathcal{O}$ be a compact, connected, orientable, irreducible, 3-orbifold of cyclic type. If $\mathcal{O}$ is topologically atoroidal, then $\mathcal{O}$ is geometric.*

In late 1981, Thurston [**Thu2, Thu6**] announced the Geometrization theorem for 3-orbifolds with non-empty ramification set (without the assumption of cyclic type), and lectured about it. Since 1986, useful notes about Thurston's proof (by Soma, Ohshika and Kojima [**SOK**] and by Hodgson [**Ho1**]) have been circulating.

In addition, in 1989 much more details appeared in Zhou's thesis [**Zh1, Zh2**] in the cyclic case. However no complete written proof was available (cf. [**Kir**, Prob. 3.46]).

Recently we have obtained with B. Leeb a proof of Thurston's orbifold theorem in the case where the singular locus has vertices. A complete written version of this proof can be found in [**BLP1, BLP2**]. This proof, in particular for orbifolds with all singular vertices of dihedral type, relies on the proof of the cyclic case presented here. However the methods used in [**BLP2**] to study the geometry of cone 3-manifolds are quite different from the ones used here.

A different proof, more in the spirit of Thurston's original approach, has been announced by D. Cooper, C. Hodgson and S. Kerckhoff in [**CHK**].

In this monograph we work in the category of orbifolds. For the basic definitions in this category, including map, homotopy, isotopy, covering and fundamental group, we refer mainly to Chapter 13 of Thurston's notes [**Thu1**], to the books by Bridson and Haefliger [**BrH**] and by Kapovich [**Kap**], as well as to the articles by Bonahon and Siebenmann [**BS1, BS2**], by Davis and Morgan [**DaM**] and by Takeuchi [**Tak1**].

In the case of good orbifolds, these notions are defined as the corresponding equivariant notions in the universal covering, which is a manifold.

According to [**BS1, BS2**] and [**Thu1**, Ch. 13], we use the following terminology.

Definitions. — We say that a compact 2-orbifold F^2 is respectively *spherical*, *discal*, *toric* or *annular* if it is the quotient by a finite smooth group action of respectively the 2-sphere S^2, the 2-disc D^2, the 2-torus T^2 or the annulus $S^1 \times [0,1]$.

A compact 2-orbifold is *bad* if it is not good. Such a 2-orbifold is the union of two non-isomorphic discal 2-orbifolds along their boundaries.

A compact 3-orbifold $\mathcal{O}$ is *irreducible* if it does not contain any bad 2-suborbifold and if every orientable spherical 2-suborbifold bounds in $\mathcal{O}$ a discal 3-suborbifold, where a *discal* 3-orbifold is a finite quotient of the 3-ball by an orthogonal action.

A connected 2-suborbifold F^2 in an orientable 3-orbifold $\mathcal{O}$ is *compressible* if either F^2 bounds a discal 3-suborbifold in $\mathcal{O}$ or there is a discal 2-suborbifold Δ^2 which intersects transversally F^2 in $\partial\Delta^2 = \Delta^2 \cap F^2$ and such that $\partial\Delta^2$ does not bound a discal 2-suborbifold in F^2.

A 2-suborbifold F^2 in an orientable 3-orbifold $\mathcal{O}$ is *incompressible* if no connected component of F^2 is compressible in $\mathcal{O}$. The compact 3-orbifold $\mathcal{O}$ is *∂-incompressible* if $\partial\mathcal{O}$ is empty or incompressible in $\mathcal{O}$.

A properly embedded 2-suborbifold $(F, \partial F) \hookrightarrow (\mathcal{O}, \partial\mathcal{O})$ is *∂-compressible* if:

- either $(F, \partial F)$ is a discal 2-suborbifold $(D^2, \partial D^2)$ which is ∂-parallel,
- or there is a discal 2-suborbifold $\Delta \subset \mathcal{O}$ such that $\partial\Delta \cap F$ is a simple arc α, $\Delta \cap \partial M$ is a simple arc β, with $\partial\Delta = \alpha \cup \beta$ and $\alpha \cap \beta = \partial\alpha = \partial\beta$

An orientable properly embedded 2-suborbifold F^2 is *∂-parallel* if it belongs to the frontier of a collar neighborhood $F^2 \times [0,1] \subset \mathcal{O}$ of a boundary component $F^2 \subset \partial\mathcal{O}$.

A properly embedded 2-suborbifold F^2 is *essential* in a compact orientable irreducible 3-orbifold, if it is incompressible, ∂-incompressible and not boundary parallel.

A compact 3-orbifold is *topologically atoroidal* if it does not contain any embedded essential orientable toric 2-suborbifold. It is *topologically acylindrical* if every properly embedded orientable annular 2-suborbifold is boundary parallel.

A *turnover* is a 2-orbifold with underlying space a 2-sphere and with three branching points. In an irreducible orientable orbifold an embedded turnover either bounds a discal 3-suborbifold or is incompressible and of non-positive Euler characteristic.

According to [**Thu1**, Ch. 13], the *fundamental group* of an orbifold $\mathcal{O}$, denoted by $\pi_1(\mathcal{O})$, is defined as the Deck transformation group of its universal cover.

A *Seifert fibration* on a 3-orbifold $\mathcal{O}$ is a partition of $\mathcal{O}$ into closed 1-suborbifolds (circles or intervals with silvered boundary) called fibres, such that each fibre has a saturated neighborhood diffeomorphic to $S^1 \times D^2/G$, where G is a finite group which acts smoothly, preserves both factors, and acts orthogonally on each factor and effectively on D^2; moreover the fibres of the saturated neighborhood correspond to the quotients of the circles $S^1 \times \{*\}$. On the boundary $\partial\mathcal{O}$, the local model of the Seifert fibration is $S^1 \times D^2_+/G$, where D^2_+ is a half disc.

A 3-orbifold that admits a Seifert fibration is called Seifert fibred. Every good Seifert fibred 3-orbifold is geometric (cf. [**Sc3**], [**Thu7**]). Seifert fibred 3-orbifolds have been classified in [**BS2**].

A compact orientable 3-orbifold $\mathcal{O}$ is *hyperbolic* if its interior is orbifold-diffeomorphic to the quotient of the hyperbolic space $\mathbb{H}^3$ by a non-elementary discrete group of isometries. In particular I-bundles over hyperbolic 2-orbifolds are hyperbolic, since their interiors are quotients of $\mathbb{H}^3$ by non-elementary Fuchsian groups. In Theorem 1, except for I-bundles, we prove that when $\mathcal{O}$ is hyperbolic, if we remove the toric components of the boundary $\partial_T\mathcal{O} \subset \partial\mathcal{O}$, then $\mathcal{O} - \partial_T\mathcal{O}$ has a hyperbolic structure with finite volume and geodesic boundary. This implies the existence of a complete hyperbolic structure on the interior of $\mathcal{O}$.

We say that a compact orientable 3-orbifold is *Euclidean* if its interior has a complete Euclidean structure. Thus, if a compact orientable and ∂-incompressible 3-orbifold $\mathcal{O}$ is Euclidean, then either $\mathcal{O}$ is a I-bundle over a 2-dimensional Euclidean closed orbifold or $\mathcal{O}$ is closed.

We say that a compact orientable 3-orbifold is *spherical* when it is the quotient of $\mathbb{S}^3$ by the orthogonal action of a finite subgroup of $SO(4)$. A spherical orbifold of cyclic type is always Seifert fibred ([**Dun1**], [**DaM**]).

Thurston's conjecture asserts that the interior of a compact irreducible orientable 3-orbifold can be decomposed along a canonical family of incompressible toric 2-suborbifolds into geometric 3-suborbifolds.