ASTÉRISQUE 283

ANALYTIC THEORY FOR THE QUADRATIC SCATTERING WAVE FRONT SET AND APPLICATION TO THE SCHRÖDINGER EQUATION

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Société Mathématique de France 2002
Publié avec le concours du Centre National de la Recherche Scientifique

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2000 *Mathematics Subject Classification.* — 35J10, 35A20, 35A27, 35A18, 35A21.

Key words and phrases. — Schrödinger equation, analytic wave front set, wave front set at infinity, propagation of singularities, FBI transform, Sjöstrand theory.

ANALYTIC THEORY FOR THE QUADRATIC SCATTERING WAVE FRONT SET AND APPLICATION TO THE SCHRÖDINGER EQUATION

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Abstract. — We consider in this work, the microlocal propagation of analytic singularities for the solutions of the Schrödinger equation with variable coefficients. We introduce, following R. Melrose and J. Wunsch, a $\mathbb{R}^n$ compactification and a cotangent compactification. We define by a FBI transform an analytic wave front set on this cotangent bundle. The main part of this paper is to prove the propagation of microlocal analytic singularities in this wave front set.

Résumé **(Théorie analytique du front d'onde de scattering quadratique et application à l'équation de Schrödinger)**

On examine dans ce travail la propagation des singularités analytiques des solutions de l'équation de Schrödinger à coefficients variables. Nous introduisons, en suivant R. Melrose et J. Wunsch, une compactification de $\mathbb{R}^n$ et une compactification du cotangent. Nous définissons sur ce cotangent un front d'onde analytique par une transformation de FBI. La majeure partie de cet article est consacrée à la preuve de la propagation des singularités analytiques microlocales de ce front d'onde.

CONTENTS

CHAPTER 0

INTRODUCTION

The purpose of this work is to provide a theory for the analytic quadratic scattering wave front set, here denoted ${}^{\mathrm{qsc}}WF_a$, which in the C^∞ case has been introduced by Wunsch [**W1**] after the work of Melrose [**M1**], and to apply it to the propagation of analytic singularities for the linear Schrödinger equation with variable coefficients.

To understand what we are doing here, let us begin by a very simple example. Let us consider the initial value problem for the constant coefficients Schrödinger equation,

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u = 0, \quad t > 0, \ \ x \in \mathbb{R}^n \\ u_{|t=0} = u_0 \end{cases} . \tag{0.1}$$

Taking $u_0 = \delta$ and $u_0 = e^{-i|x|^2}$, it is an easy exercise to see that a data which is a distribution with compact support may give rise to a smooth solution (in x) for every positive t, while an analytic data which oscillates at infinity may produce a singular solution (in x) at some time t. This classical fact, which, roughly speaking, asserts that the smoothness of the solution (in x), for $t > 0$, is under the control of the behavior at infinity of the initial data, is known as "propagation with infinite speed".

It turns out that this fact extends in many directions. It is of microlocal nature, it can be described geometrically and it holds for non trapping Laplacians which are flat perturbation (at infinity) of the constant coefficient case.

These extensions have been the subject of many recent works. See Kapitanski-Safarov [**KS**], Craig-Kappeler-Strauss [**CKS**], Craig [**C**], Shananin [**Sh**], Robbiano-Zuily [**RZ1, RZ2**], Kajitani-Wakabayashi [**KW**], Okaji [**O**], Morimoto-Robbiano-Zuily [**MRZ**]. Related works have been done by Doi [**D1, D2**], Hayashi-Kato [**HK**], Hayashi-Saitoh [**HS**], Kajitani [**K**], Vasy [**V**], Vasy-Zworski [**VZ**] and we refer to the paper [**CKS**] for a more complete bibliography.

In all these works we are handling two informations : behavior at infinity (decay, oscillations...) and smoothness. In a recent paper, Wunsch [**W1**] proposed to embed these two informations in one unique object, which he called the C^∞ quadratic scattering (qsc) wave front set, in which the above phenomena of infinite speed propagation would appear as a propagation of singularities result. Here the word quadratic is used to emphasize that this wave front set takes in account the quadratic oscillations at infinity. Let us note that a scattering wave front set in the C^∞ case was already introduced by Melrose [**M1, M2**] and that related notions have been recently considered by Wunsch-Zworski [**WZ**] (see also Rouleux [**R**]). Moreover, in the same paper Wunsch gave a quite complete description of the propagation of singularities for this C^∞ wave front set which will be described later one.

It is worthwhile to mention that some propagation results have been obtained a long time ago by R. Lascar [**L**] (see also Boutet de Monvel [**B**]). In the C^∞ case, he introduced a parabolic wave front set and he proved its propagation. However this propagation (in x) holds between two points at the same time t ; it is therefore unable to link the "singularities" of the data to those of the solution for positive time.

The work of Wunsch relies on some geometrical point of view of Melrose. It begins by working on a compact manifold M with boundary ∂M, which comes from a (stereographic) compactification of $\mathbb{R}^n$. Roughly speaking this corresponds to set, for large x, $x = \omega/\rho$, where $\rho > 0$ and $\omega \in S^{n-1}$. The boundary ∂M corresponds then to the infinity of $\mathbb{R}^n$. The second step is to define a cotangent bundle. The natural one, coming from the above compactification would be the one where the canonical one form is given by $\alpha = \lambda \frac{d\rho}{\rho^2} + \mu \cdot \frac{dy}{\rho}$ if (ρ, y) are local coordinates near the boundary. However, having in mind that this bundle should hold the singularities of the quadratic oscillatory data, Wunsch introduced the quadratic scattering (qsc) cotangent bundle, ${}^{\mathrm{qsc}}T^*M$ where the canonical one form is given by $\alpha = \lambda \frac{d\rho}{\rho^3} + \mu \cdot \frac{dy}{\rho^2}$. Indeed if $u_0(x) = e^{i\langle Ax,x\rangle}$, where A is an $n \times n$ symmetric real matrix, we have $u_0 = e^{\frac{i}{\rho^2}\langle A\omega,\omega\rangle}$ and the differential of the phase is

$$d\left(\frac{1}{\rho^2}\langle A\omega,\omega\rangle\right) = -2\langle A\omega,\omega\rangle\frac{d\rho}{\rho^3} + \sum_{j=1}^{n}\frac{\partial}{\partial\omega_j}(\langle A\omega,\omega\rangle)\frac{d\omega_j}{\rho^2}\,.$$

Local coordinates, near the boundary, in this qsc cotangent bundle are given by (ρ, y, λ, μ). Now, since only high frequencies are involved in the occurring of singularities, Melrose suggests to make a radial compactification in the fibers, that is to set, for large $\lambda + |\mu|$,

$$\sigma = \frac{1}{(\lambda^2 + |\mu|^2)^{1/2}}, \quad \overline{\lambda} = \sigma\lambda, \quad \overline{\mu} = \sigma\mu\,.$$

Then we may define the extended qsc cotangent bundle ${}^{\mathrm{qsc}}\overline{T}^*M$ in which local coordinates, near the boundary of M, are given by $(\rho, y, \sigma, (\overline{\lambda}, \overline{\mu}))$, where $\rho \geqslant 0$, $\sigma \geqslant 0$. Its boundary $\mathcal{C}$ is the union of two faces, ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M = \{(\rho, y, \sigma, (\overline{\lambda}, \overline{\mu})) : \rho = 0\}$ and ${}^{\mathrm{qsc}}S^*M = \{(\rho, y, \sigma, (\overline{\lambda}, \overline{\mu})) : \sigma = 0\}$.

The qsc wave front set is a subset of $\mathcal{C}$. To define it, in the C^∞ case, Wunsch uses Melrose's theory of pseudo-differential operators on manifolds with corners [**M1**]. Here, in the analytic case (but also in the C^∞ or Gevrey cases) we use instead the Sjöstrand machinery of FBI transforms. Our analytic qsc wave front set will be defined through a FBI transform with two scales (h,k), instead of only one scale $\lambda = 1/k$ in the usual case. More precisely we shall set for $u \in L^2(M)$,

$$(0.2) \quad \mathcal{T}u(\alpha,h,k) = \iint e^{ih^{-2}k^{-1}\varphi(\rho/h,y,\alpha,h)} a(\rho/h,y,\alpha,h,k)\chi(\rho/h,y)\overline{u(\rho,y)}\,d\rho dy\,.$$

Here φ is a phase, a a symbol and χ a cut-off function. (See § 2 for the precise definitions of phases, symbols and ${}^{\mathrm{qsc}}WF_a$).

The simplest phase is the following

$$\varphi(s,y,\alpha,h) = (s-\alpha_s)\alpha_\tau + (y-\alpha_y)\cdot\alpha_\eta + ih[(s-\alpha_s)^2 + (y-\alpha_y)^2]\,,$$

where $\alpha = (\alpha_s,\alpha_y,\alpha_\tau,\alpha_\eta) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1}$.

Now, if $u(t,\cdot)$ is a solution of (0.1) and $t_0 > 0$, the ${}^{\mathrm{qsc}}WF_a(u(t_0,\cdot))$ does not propagate ; instead we introduce a uniform qsc analytic wave front set ${}^{\mathrm{qsc}}\widetilde{WF}_a(u(t_0,\cdot))$ which will propagate.

In (0.2), the parameter h is used to describe the behavior at infinity (decay, oscillations. . .) while k is used to test the analytic smoothness. However near the corner $\{\rho = \sigma = 0\}$ these two informations are mixed. As in the usual case, it is necessary to define such transforms for a large class of phases. Moreover one should be able to change phases, symbols and cut-off functions, in particular, to show the invariance of the ${}^{\mathrm{qsc}}WF_a$; to achieve these invariances, in particular to go from one phase to another, one has to make a careful study of the pseudo-differential operators in the complex domain, then in the real domain and to pass from the first theory to the second by some delicates changes of contours. Here the situation is complicated by the fact that our FBI phases have an imaginary part which goes to zero with h. In the appendix the reader will find a complete Sjöstrand's theory in the case of two scales.

Concerning the propagation theorems we consider a Schrödinger equation with a Laplacian Δ_g with respect to a scattering metric g in the sense of Melrose ; this means that, near the boundary one can write $g = \frac{d\rho^2}{\rho^4} + \frac{h}{\rho^2}$, where h is a metric such that $h|_{\partial M}$ is positive definite. This includes, of course the flat metric for which $h = d\omega^2$, but also the asymptotically flat metrics on $\mathbb{R}^n$. In this setting we try to answer the following question. Let m_0 be a point in $\mathcal{C} = {}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M \cup {}^{\mathrm{qsc}}S^*M$, u be a solution of the initial value problem for this Schrödinger equation and $T > 0$. On what condition on u_0 do we have $m_0 \notin {}^{\mathrm{qsc}}WF_a(u(T,\cdot))$? The answer, which depends strongly on the position of m_0 in $\mathcal{C}$, requires a careful study of the flow of the Laplacian on $\mathcal{C}$. This can be found in Wunsch [**W1**] ; however a still more precise description near the corner $\{\rho = \sigma = 0\}$ is needed here. The different statements, according to the position of m_0 in $\mathcal{C}$, will follow from four propagation results : propagation inside ${}^{\mathrm{qsc}}T^*_{\partial M}M$, inside ${}^{\mathrm{qsc}}S^*M$ or along the corner (for the uniform ${}^{\mathrm{qsc}}WF_a$ and fixed t), from the interior to the corner and finally from the boundary at infinity to the corner. To give a flavor of the results obtained, let us describe the case of the first situation. Let $0 \leqslant t_1 < t_2$ and $m_0 \in {}^{\mathrm{qsc}}\overline{T}^*M$. Assume that $\exp tH_\Delta(m_0)$ (the flow of the Laplacian at infinity through m_0) stays, for $t \in [t_1, t_2]$, inside the interior of ${}^{\mathrm{qsc}}\overline{T}^*_{\partial M}M$. Then $\exp t_1 H_\Delta(m_0)$ does not belong to ${}^{\mathrm{qsc}}WF_a(u(t_1,\cdot))$ if and only if $\exp t_2 H_\Delta(m_0)$ does not belong to ${}^{\mathrm{qsc}}WF_a(u(t_2,\cdot))$. Coming back to the above question, this result can be applied (with $t_1 = 0, t_2 = t$) when $m_0 = (0, y_0, \lambda_0, \mu_0)$ in the following cases : $\mu_0 \neq 0$ or $\mu_0 = 0$, $\lambda_0 > 0$ or $\mu_0 = 0$, $\lambda_0 < 0$, $t < -1/2\lambda_0$, because, in the later case, the flow starting from m_0 reaches the corner after a finite time $t = -1/2\lambda_0$.

A complete description of the other cases can be found in § 4.

Let us now describe the method of proofs. The first idea, which comes from Sjöstrand's work [**Sj**], is that the FBI transform can be used, at the same time, to test the microlocal smoothness and, as a Fourier integral operator, to reduce an operator to a simpler form. Let us be more precise. We look for a family of phases $\varphi = \varphi(\theta; \rho/h, y, \alpha, h)$ and symbols $a = a(\theta; \rho/h, y, \alpha, h, k)$ depending on a parameter θ, such that

$$\left(\frac{\partial}{\partial\theta} + i\Delta_g^*\right)(ae^{ih^{-2}k^{-1}\varphi}) = \mathcal{O}\left(e^{-\varepsilon/hk}\right), \quad \varepsilon > 0\,, \tag{0.3}$$

where Δ_g^* is the adjoint of the Laplacian Δ_g.

This leads to the eikonal equation for φ,

$$\frac{\partial\varphi}{\partial\theta} + p\left(sh, y, s^2\frac{\partial\varphi}{\partial s}, s\frac{\partial\varphi}{\partial y}\right) = 0 \tag{0.4}$$

and to the transport equations,

$$Xa_j + h^2kQa_{j-1} = b_j\,, \quad \text{if} \quad a = \sum_{j=0}^{+\infty}(h\sqrt{k})^j a_j\,, \tag{0.5}$$

where X is a non degenerate real vector field and Q a second order differential operator.

As soon as we have solved these equations, we see that the corresponding FBI transform $\mathcal{T}u(\theta; t, \alpha, h, k)$ satisfies the real transport equation

$$\left(\frac{1}{k}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial t}\right)\mathcal{T}u(\theta; t, \alpha, h, k) = \mathcal{O}\left(e^{-c/hk}\right), \quad c > 0\,, \tag{0.6}$$

and the propagation theorems follow easily.

The main point of the paper is therefore to solve (0.4) and (0.5). The resolution of the eikonal equation (0.4) requires the use of the complex symplectic geometry. We make a careful study of the bicharacteristic flow to span a nice complex Lagrangian manifold on which the symbol $q = \theta^* + p(sh, y, \tau s^2, s\eta)$ vanishes. It should be noted