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**TWO-DIMENSIONAL MARKOVIAN
HOLONOMY FIELDS**

Thierry LÉVY

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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TWO-DIMENSIONAL MARKOVIAN HOLONOMY FIELDS

Thierry Lévy

Abstract. — This text defines and studies a class of stochastic processes indexed by curves drawn on a compact surface and taking their values in a compact Lie group. We call these processes two-dimensional Markovian holonomy fields. The prototype of these processes, and the only one to have been constructed before the present work, is the canonical process under the Yang-Mills measure, first defined by Ambar Sengupta and later by the author. The Yang-Mills measure sits in the class of Markovian holonomy fields very much like the Brownian motion in the class of Lévy processes. We prove that every regular Markovian holonomy field determines a Lévy process of a certain class on the Lie group in which it takes its values, and we construct, for each Lévy process in this class, a Markovian holonomy field to which it is associated. When the Lie group is in fact a finite group, we give an alternative construction of this Markovian holonomy field as the monodromy of a random ramified principal bundle. Heuristically, this agrees with the physical origin of the Yang-Mills measure as the holonomy of a random connection on a principal bundle.

Résumé. (Champs d'holonomie markoviens bidimensionnels). — Ce travail est consacré à la définition et à l'étude d'une classe de processus stochastiques indexés par des chemins tracés sur une surface, qui prennent leurs valeurs dans un groupe de Lie compact et qui satisfont une propriété d'indépendance conditionnelle analogue à la propriété de Markov. Nous appelons ces processus des champs d'holonomie markoviens bidimensionnels. L'exemple fondamental de cette sorte de processus est le processus canonique sous la mesure de Yang-Mills, qui a été construite d'abord par Ambar Sengupta puis plus tard par l'auteur. C'est aussi le seul champ d'holonomie markovien qui ait été construit avant ce travail. Le processus canonique sous la mesure de Yang-Mills est assez exactement aux champs d'holonomie markoviens ce que le mouvement brownien est aux processus de Lévy. Deux de nos principaux résultats affirment qu'à tout champ d'holonomie markovien suffisamment régulier est associé un processus de Lévy d'une certaine classe sur le groupe de Lie dans lequel il prend ses valeurs et réciproquement que pour tout processus de Lévy dans cette classe il existe un champ d'holonomie markovien auquel il est associé. Dans le cas particulier où le groupe de

Lie considéré est un groupe fini, nous parvenons à réaliser ce champ d'holonomie markovien comme la monodromie d'un fibré principal ramifié aléatoire. Ceci nous rapproche de l'interprétation originelle de la mesure de Yang-Mills, issue de la théorie quantique des champs, comme mesure de probabilités sur l'espace des connexions sur un fibré principal.

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INTRODUCTION

The elementary theory of Markov processes establishes a correspondence between several types of objects among which transition semigroups and stochastic processes. These stochastic processes can take their values in fairly general spaces, but they are usually indexed by a subset of the real numbers, for the Markov property relies on the distinction between past and future. In the present work, we investigate a correspondence between certain transition semigroups and another kind of stochastic processes, where the notions of past and future are replaced by the notions of inside and outside. The processes that we consider are indexed by curves, or rather loops, drawn on a surface, and they take their values in a compact Lie group. We call them (2-dimensional) *Markovian holonomy fields*. They are Markovian in the following sense: if some piece of a surface is bounded by a finite collection of loops, then the values of the process on loops located inside this piece and outside this piece are independent given the value of the process on the finite collection of loops which bound this piece.

0.1. A 1-dimensional analogue

Let us start by discussing the 1-dimensional analogues of Markovian holonomy fields, which are just Markov processes looked at from a slightly unusual point of view. Let us choose a transition semigroup $P = (P_t)_{t \geq 0}$ on some state space \mathcal{X} . For each $t \geq 0$, $P_t(x, dy)$ is a transition kernel on $\mathcal{X} \times \mathcal{X}$. Under suitable assumptions, we can associate to P a homogeneous Markov process with values in \mathcal{X} , which we denote by X . This Markov process is not really a single stochastic process, it is rather a collection of processes, essentially one for each initial condition at a specific time. In fact, if we consider X restricted to segments, we can say that to each segment $[a, b] \subset \mathbb{R}$ and each initial condition $x \in \mathcal{X}$ we associate a process $(X_t)_{t \in [a, b]}$ with values in \mathcal{X} such that $X_a = x$ almost surely. Within the structure implied by the fact that $[a, b]$ is a subset of \mathbb{R} , what we really use is the topological structure of this interval, its orientation and our ability to measure the distance between any two of its points. Of course, in the present 1-dimensional setting, this structure suffices to characterise

the interval up to translation, and the last sentence may seem pointless. Its content should however become clearer in the 2-dimensional setting.

Let us push the abstraction a little further and try to define, for all compact 1-dimensional manifold M , a process $(X_t)_{t \in M}$ with values in \mathcal{X} . As we have just observed, we need an orientation of M and a way to measure distances. If M is not connected, let us agree that the restrictions of our process to the various connected components of M will be independent. So, let M be a connected oriented compact Riemannian 1-dimensional manifold. There are not so many options: M is either homeomorphic to a segment or to a circle, it has a certain positive total length, and this information characterises it completely up to orientation-preserving isometry. If M is a segment of length L , it is isometric to $[0, L]$ and there is no difficulty in defining the process $(X_t)_{t \in M}$ given an initial condition. Before turning to the case of the circle, let us interpret the Markov property of X in terms of these 1-dimensional manifolds.

Let M_1 and M_2 be two manifolds as above, isometric to segments. Let $M_1 \cdot M_2$ denote the manifold obtained by identifying the final point of M_1 with the initial point of M_2 . It is still homeomorphic to a segment. Choose an initial condition $x \in \mathcal{X}$. We are able to construct two stochastic processes indexed by $M_1 \cdot M_2$. On one hand, we can take $M_1 \cdot M_2$ as a segment on its own and simply consider the process $(X_t)_{t \in M_1 \cdot M_2}$ with initial condition x . On the other hand, we can also proceed as follows. For all segment M and all $x \in \mathcal{X}$, let $\mathcal{L}(x, M)$ denote the distribution of the process $(X_t)_{t \in M}$ with initial condition x . Let us also denote by $\mathcal{L}(x, M, dy)$ the disintegration of $\mathcal{L}(x, M)$ with respect to the value of X at the final point of M . Then the probability measure $\int_{\mathcal{X}} \mathcal{L}(x, M_1, dy) \otimes \mathcal{L}(y, M_2)$ is the distribution of a process indexed by the disjoint union of M_1 and M_2 which takes the same value at the final point of M_1 and the initial point of M_2 . It can thus be identified with the distribution of a process indexed by $M_1 \cdot M_2$. It is exactly the content of the Markov property of X that the two measures that we have considered are equal:

$$(1) \quad \int_{\mathcal{X}} \mathcal{L}(x, M_1, dy) \otimes \mathcal{L}(y, M_2) = \mathcal{L}(x, M_1 \cdot M_2).$$

This example illustrates in the simplest possible way the fact that the Markov property can be nicely formulated in terms of surgery of manifolds, in this case in terms of concatenation of intervals. Manifolds of dimension 1 undergo another kind of surgery, when the two endpoints of a single interval are glued together (see Figure 1). If we try to mimic (1), we are tempted to define the distribution of a process indexed by the circle S^1 of length L , seen as the interval $[0, L]$ of which the endpoints have been