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Pablo Pelaez

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Mots-clefs. — Filtration par les tranches, motifs mixtes, suite spectrale motivique de Atiyah-Hirzebruch, théorie homotopique des schémas.

MULTIPLICATIVE PROPERTIES OF THE SLICE FILTRATION

Pablo PELAEZ

Abstract. — Let S be a Noetherian separated scheme of finite Krull dimension, and $\mathscr{SH}(S)$ be the motivic stable homotopy category of Morel-Voevodsky. In order to get a motivic analogue of the Postnikov tower, Voevodsky [**25**] constructs the slice filtration by filtering $\mathscr{SH}(S)$ with respect to the smash powers of the multiplicative group $\mathbb{G}_m$. We show that the slice filtration is compatible with the smash product in Jardine's category $\mathrm{Spt}_T^{\Sigma}\mathscr{M}_*$ of motivic symmetric T-spectra [**14**], and describe several interesting consequences that follow from this compatibility. Among them, we have that over a perfect field all the slices s_q are in a canonical way modules in $\mathrm{Spt}_T^{\Sigma}\mathscr{M}_*$ over the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$, and if the field has characteristic zero it follows that the slices s_q are big motives in the sense of Voevodsky, this relies on the work of Levine [**16**], Röndigs-Østvær [**22**] and Voevodsky [**26**]. It also follows that the smash product in $\mathrm{Spt}_T^{\Sigma}\mathscr{M}_*$ induces pairings in the motivic Atiyah-Hirzebruch spectral sequence.

Résumé **(Les propriétés multiplicatives de la filtration par les tranches).** — Soit S un schéma noethérien séparé de dimension de Krull finie, et $\mathscr{SH}(S)$ la catégorie homotopique stable de Morel-Voevodsky. Afin d'obtenir un analogue motivique de la tour de Postnikov, Voevodsky [**25**] définit la filtration par les tranches dans $\mathscr{SH}(S)$ considérant les smash-produits itérées de le groupe multiplicatif $\mathbb{G}_m$. Nous montrons que la filtration par les tranches est compatible avec le smash-produit dans la catégorie de Jardine $\mathrm{Spt}_T^{\Sigma}\mathscr{M}_*$ des T-spectres symétriques motiviques [**14**]. Cette compatibilité a plusieurs conséquences intéressantes. D'entre eux, sur un corps parfait tous les tranches s_q sont canoniquement modules dans $\mathrm{Spt}_T^{\Sigma}\mathscr{M}_*$ sur le spectre motivique d'Eilenberg-MacLane $H\mathbb{Z}$, et si le corps est de charactéristique zéro les tranches s_q sont motifs grands au sens de Voevodsky, ce utilise les résultats de Levine [**16**], Röndigs-Østvær [**22**] et Voevodsky [**26**]. Nous montrons aussi que le smash-produit dans $\mathrm{Spt}_T^{\Sigma}\mathscr{M}_*$ induit des structures multiplicatives sur la suite spectrale motivique de Atiyah-Hirzebruch.

CONTENTS

INTRODUCTION

Let S be a Noetherian separated scheme of finite Krull dimension, $\mathcal{S}m|_S$ be the category of smooth schemes of finite type over S, $\mathcal{M}_*$ be the category of pointed simplicial presheaves on $\mathcal{S}m|_S$ equipped with the motivic model structure of Morel and Voevodsky [18], and T in $\mathcal{M}_*$ be $S^1 \wedge \mathbb{G}_m$ where $\mathbb{G}_m$ is the multiplicative group $\mathbb{A}^1_S - \{0\}$ pointed by 1, and S^1 is the simplicial circle. Let $\mathrm{Spt}_T\mathcal{M}_*$ denote the category of T-spectra on $\mathcal{M}_*$ equipped with the motivic stable model structure, and $\mathcal{SH}(S)$ its homotopy category, which is triangulated. We consider the following objects in $\mathrm{Spt}_T\mathcal{M}_*$:

$$C^q_{\text{eff}} = \{F_n(S^r \wedge \mathbb{G}^s_m \wedge U_+) \mid n, r, s \geq 0; s - n \geq q; U \in \mathcal{S}m|_S\}$$

where F_n is the left adjoint to the n-evaluation functor $ev_n : \mathrm{Spt}_T\mathcal{M}_* \to \mathcal{M}_*$. In order to get a motivic version of the Postnikov tower, Voevodsky [25] constructs a filtered family of triangulated subcategories of $\mathcal{SH}(S)$, which we call the *slice filtration*:

$$\cdots \subseteq \Sigma^{q+1}_T \mathcal{SH}^{\text{eff}}(S) \subseteq \Sigma^q_T \mathcal{SH}^{\text{eff}}(S) \subseteq \Sigma^{q-1}_T \mathcal{SH}^{\text{eff}}(S) \subseteq \cdots \tag{1}$$

where $\Sigma^q_T \mathcal{SH}^{\text{eff}}(S)$ is the smallest full triangulated subcategory of $\mathcal{SH}(S)$ which contains C^q_{eff} and is closed under arbitrary coproducts. The work of Neeman [19], [20], shows that the inclusion:

$$i_q : \Sigma^q_T \mathcal{SH}^{\text{eff}}(S) \hookrightarrow \mathcal{SH}(S)$$

has a right adjoint r_q, and that the functors:

$$f_q, s_q : \mathcal{SH}(S) \to \mathcal{SH}(S)$$

are exact, where $f_q = i_q r_q$, and for every X in $\mathrm{Spt}_T\mathcal{M}_*$, $s_q(X)$ fits in the following distinguished triangle:

$$f_{q+1}X \xrightarrow{\rho^X_q} f_q X \xrightarrow{\pi^X_q} s_q X \longrightarrow \Sigma^{1,0}_T f_{q+1} X$$

We say that $f_q(X)$ is the $(q-1)$-connective cover of X and $s_q(X)$ the q-slice of X.

Let $\mathrm{Spt}^\Sigma_T\mathcal{M}_*$ denote the category of symmetric T-spectra on $\mathcal{M}_*$ equipped with the motivic symmetric stable model structure defined by Jardine [14], and $\mathcal{SH}^\Sigma(S)$ its homotopy category, which is triangulated. Since the adjunction:

$$(V, U, \varphi) : \mathrm{Spt}_T\mathcal{M}_* \to \mathrm{Spt}^\Sigma_T\mathcal{M}_*$$

given by the symmetrization and forgetful functors is a Quillen equivalence [**14**], we get a similar filtration for $\mathcal{SH}^{\Sigma}(S)$. Using the slice filtration (1), it is possible to construct a spectral sequence which is an analogue of the classical Atiyah-Hirzebruch spectral sequence in algebraic topology. Namely, let X, Y be in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$, and $[-,-]_{\mathrm{Spt}}^{\Sigma}$ be the set of maps between two objects in $\mathcal{SH}^{\Sigma}(S)$. Then the collection of distinguished triangles:

$$\{f_{q+1}^{\Sigma}X \to f_q^{\Sigma}X \to s_q^{\Sigma}X \to \Sigma_T^{1,0} f_{q+1}^{\Sigma}X\}$$

generates an exact couple $(D_1^{p,q}(Y;X), E_1^{p,q}(Y;X))$, where:

1. $D_1^{p,q}(Y;X) = [Y, \Sigma_T^{p+q,0} f_p^{\Sigma}X]_{\mathrm{Spt}}^{\Sigma}$, and
2. $E_1^{p,q}(Y;X) = [Y, \Sigma_T^{p+q,0} s_p^{\Sigma}X]_{\mathrm{Spt}}^{\Sigma}$.

Let A be a cofibrant ring spectrum with unit in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$, and A-mod be the category of left A-modules in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$. It follows directly from the work of Jardine [**14**] and Hovey [**9**] that the adjunction:

$$(A \wedge -, U, \varphi) : \mathrm{Spt}_T^{\Sigma}\mathcal{M}_* \to A\text{-mod}$$

induces a model structure A-mod$(\mathcal{M}_*)$ in A-mod, i.e. a map f in A-mod$(\mathcal{M}_*)$ is a weak equivalence or a fibration if and only if Uf is a weak equivalence or a fibration in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$. In the rest of this introduction $p, q \in \mathbb{Z}$ will denote arbitrary integers, and $u : \mathbf{1} \to A$ the unit map of A.

Our main results are the following:

1. There exists a model structure $R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$ for symmetric T-spectra on $\mathcal{M}_*$ such that its homotopy category $R_{C_{\mathrm{eff}}^q}\mathcal{SH}^{\Sigma}(S)$ is triangulated and naturally equivalent as a triangulated category to $\Sigma_T^q\mathcal{SH}^{\mathrm{eff}}(S)$ (see diagrams (2) and (3) at the end of this introduction). Furthermore, the identity:
$$\mathrm{id} : \mathrm{Spt}_T^{\Sigma}\mathcal{M}_* \to R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$$
is a right Quillen functor, and the functor f_q is canonically isomorphic to the following composition of exact functors:
$$\mathcal{SH}^{\Sigma}(S) \xrightarrow{R_{\Sigma}} R_{C_{\mathrm{eff}}^q}\mathcal{SH}^{\Sigma}(S) \xrightarrow{C_q^{\Sigma}} \mathcal{SH}^{\Sigma}(S)$$
where R_{Σ} denotes a fibrant replacement functor in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$, and C_q^{Σ} denotes a cofibrant replacement functor in $R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$. For the proof the reader may look at Theorem 3.3.9, Corollary 3.3.17, Theorem 3.3.25, corollary 3.3.5, and Theorem 3.3.22.
2. There exists a model structure $S^q\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$ for symmetric T-spectra on $\mathcal{M}_*$ such that its homotopy category $S^q\mathcal{SH}^{\Sigma}(S)$ is triangulated and the identity:
$$\mathrm{id} : R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T^{\Sigma}\mathcal{M}_* \to S^q\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$$