337 ASTÉRISQUE

2011

PERIODIC TWISTED COHOMOLOGY AND T-DUALITY

Ulrich Bunke & Thomas Schick & Markus Spitzweck

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque est un périodique de la Société mathématique de France.

Numéro 337, 2011

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 30 € ($ 45)

Abonnement Europe : 463 €, hors Europe : 512 € ($ 768)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0303-1179

ISBN 978-2-85629-307-2

Directeur de la publication : Aline BONAMI

337

ASTÉRISQUE

2011

PERIODIC TWISTED COHOMOLOGY AND T-DUALITY

Ulrich Bunke & Thomas Schick & Markus Spitzweck

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Ulrich Bunke
Mathematische Fakultät
Universität Regensburg
93040 Regensburg, Germany
ulrich.bunke@mathematik.uni-regensburg.de

Thomas Schick
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen, Germany
schick@uni-math.gwdg.de

Markus Spitzweck
Mathematische Fakultät
Universität Regensburg
93040 Regensburg, Germany
markus.spitzweck@mathematik.uni-regensburg.de

***Classification mathématique par sujet* (2000).** — 55N30, 46M20, 14A20.

Mots-clefs. — Cohomologie tordue, Verdier dualité, stack topologique, théorie des faisceaux, T-dualité, orbi-espaces, categorie derivée non-bornée, cohomologie de de Rham tordue.

PERIODIC TWISTED COHOMOLOGY AND T-DUALITY

Ulrich BUNKE, Thomas SCHICK and Markus SPITZWECK

Abstract. — Using the differentiable structure, twisted 2-periodic de Rham cohomology is well known, and showing up as the target of Chern characters for twisted K-theory. The main motivation of this work is a topological interpretation of two-periodic twisted de Rham cohomology which is generalizable to arbitrary topological spaces and at the same time to arbitrary coefficients.

To this end we develop a sheaf theory in the context of locally compact topological stacks with emphasis on:

— the construction of the sheaf theory operations in unbounded derived categories
— elements of Verdier duality
— and integration.

The main result is the construction of a functorial periodization associated to a $U(1)$-gerbe.

As an application we verify the T-duality isomorphism in periodic twisted cohomology and in periodic twisted orbispace cohomology.

***Résumé* (Cohomologie périodique tordue et T-dualité).** — La cohomologie de de Rham tordue (periodique de période 2) est une construction bien connue, elle est importante en tant que codomaine d'un caractère de Chern pour la K-theorie tordue.

La motivation principale de notre livre est une interprétation topologique de la cohomologie de de Rham tordue, une interprétation avec géneralisations á des espaces et coefficients arbitraires.

Dans ce but, nous développons une théorie des faisceaux sur des piles topologiques localement compactes, et plus particulièrement :

— la construction des opérations de la théorie des faisceaux dans les catégories derivées non-bornées,
— les élements de la dualité de Verdier,
— et l'integration.

Notre résultat principal est la construction d'une périodisation fonctorielle associé a une $U(1)$-gerbe.

Parmi les applications, citons la vérification d'un isomorphisme de T-dualité pour la cohomologie périodique tordue et celle des orbi-espaces.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Periodic twisted cohomology

1.1.1. — The twisted de Rham cohomology $H_{dR}(M,\omega)$ of a manifold M equipped with a closed three form $\omega \in \Omega^3(M)$ is the two-periodic cohomology of the complex

$$\Omega(M,\omega)_{\mathrm{per}} \colon \cdots \to \Omega^{\mathrm{ev}}(M) \stackrel{d_\omega}{\to} \Omega^{\mathrm{odd}}(M) \stackrel{d_\omega}{\to} \Omega^{\mathrm{ev}}(M) \to \cdots \ , \tag{1.1.1}$$

where $d_\omega := d_{dR} + \omega$ is the sum of the de Rham differential and the operation of taking the wedge product with the form ω. The two-periodic twisted de Rham cohomology is interesting as the target of the Chern character from twisted K-theory [**1**], [**19**], [**3**], or as a cohomology theory which admits a T-duality isomorphism [**4**], [**7**].

1.1.2. — In [**9**] we developed a sheaf theory for smooth stacks. Let $f \colon G \to X$ be a gerbe with band $U(1)$ over a smooth stack X, and consider a closed three-form $\omega \in \Omega^3_X(X)$ which represents the image of the Dixmier-Douady class of the gerbe $G \to X$ in de Rham cohomology. The main result of [**9**] states that there exists an isomorphism

$$Rf_* f^* \underline{\mathbb{R}}_{\mathbf{X}} \stackrel{\sim}{\longleftarrow} \Omega_X[[z]]_\omega \tag{1.1.2}$$

in the bounded below derived category $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ of sheaves of abelian groups on X. Here $\underline{\mathbb{R}}_{\mathbf{X}}$ denotes the constant sheaf with value $\mathbb{R}$ on X. Furthermore, $\Omega_X[[z]]_\omega$ is the sheaf of formal power series of smooth forms on X, where $\deg(z) = 2$, and its differential is given by $d_\omega := d_{dR} + \omega\frac{d}{dz}$. The isomorphism is not canonical, but depends on the choice of a connection on the gerbe G with characteristic form ω.

1.1.3. — The complex (1.1.1) can be defined for a smooth stack X equipped with a three-form $\omega \in \Omega^3_X(X)$. It is the complex of global sections of a sheaf of two-periodic complexes $\Omega_{X,\omega,\mathrm{per}}$ on X. The complex of sheaves $\Omega_X[[z]]_\omega$ is not two-periodic. The relation between $\Omega_X[[z]]_\omega$ and $\Omega_{X,\omega,\mathrm{per}}$ has been discussed in [**9**, 1.3.23]. Consider the diagram

$$\mathcal{D} \colon \Omega(X)[[z]]_\omega \stackrel{\frac{d}{dz}}{\leftarrow} \Omega(X)[[z]]_\omega \stackrel{\frac{d}{dz}}{\leftarrow} \Omega(X)[[z]]_\omega \stackrel{\frac{d}{dz}}{\leftarrow} \cdots \ . \tag{1.1.3}$$

Then there exists an isomorphism

$$\Omega_{X,\omega,\mathrm{per}} \cong \operatorname{holim} \mathscr{D} \ . \tag{1.1.4}$$

1.1.4. — As mentioned above, the isomorphism (1.1.2) depends on the choice of a connection on the gerbe G. Moreover, the diagram $\mathscr{D}$ depends on these choices via ω. In order to construct a natural two-periodic cohomology one must find a natural replacement of the operation $\frac{d}{dz}$ which acts on the left-hand side $Rf_*f^*\underline{\mathbb{R}}_{\mathbf{X}}$ of (1.1.2). It is the first goal of this paper to carry this out properly.

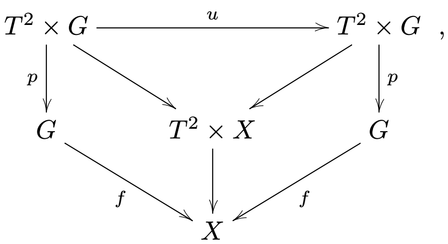
1.1.5. — One can do this construction in the framework of smooth stacks developed in [**9**]. But for the present paper we choose the setting of topological stacks. Only in Subsection 2.3 we work in smooth stacks and discuss the connection with [**9**]. In Section 6 we develop some aspects of the theory of locally compact stacks and the sheaf theory in this context. For the purpose of this introduction we freely use notions and constructions from this theory. We hope that the ideas are understandable by analogy with the usual case of sheaf theory on locally compact spaces.

1.1.6. — Let $G \to X$ be a $U(1)$-banded gerbe over a locally compact stack. The main object of the present paper is a periodization functor

$$P_G : D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$

which is functorial in $G \to X$, and where $D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ and $D(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$ denote the bounded below and unbounded derived categories of sheaves of abelian groups on the site $\mathbf{X}$ of the stack X. A simple construction of the isomorphism class of $P_G(F)$ is given in Definition 2.4.2. The functorial version is much more complicated. Its construction is completed in Definition 3.4.5.

1.1.7. — Let us sketch the construction of P_G. Recall that gerbes with band $U(1)$ over a locally compact stack Y are classified by $H^3(Y;\mathbb{Z})$, and automorphisms of a given $U(1)$-gerbe are classified by $H^2(Y;\mathbb{Z})$ [**14**]. We consider the diagram



where the automorphism u of gerbes over $T^2 \times X$ is classified by $\mathrm{or}_{T^2} \times 1 \in H^2(T^2 \times X;\mathbb{Z})$, and where or_{T^2} denotes the orientation class of the two-torus. We define a natural transformation

$$D\colon Rf_*f^* \to Rf_*f^* \colon D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X}) \to D^+(\mathrm{Sh}_{\mathrm{Ab}}\mathbf{X})$$