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Abstract. — A class of pseudodifferential operators on the Heisenberg group is defined. As it should be, this class is an algebra containing the class of differential operators. Furthermore, those pseudodifferential operators act continuously on Sobolev spaces and the loss of derivatives may be controled by the order of the operator. Although a large number of works have been devoted in the past to the construction and the study of algebras of variable-coefficient operators, including some very interesting works on the Heisenberg group, our approach is different, and in particular puts into light microlocal directions and completes, with the Littlewood-Paley theory initiated in 2000 by Bahouri, Gérard and Xu, a microlocal analysis of the Heisenberg group.

***Résumé* (Analyse dans l'espace des phases, et calcul pseudodifférentiel sur le groupe de Heisenberg).** — Nous définissons une classe d'opérateurs pseudo-différentiels sur le groupe de Heisenberg. Comme il se doit, cette classe constitue une algèbre contenant les opérateurs différentiels. De plus, ces opérateurs pseudo-différentiels sont continus sur les espaces de Sobolev et l'on peut contrôler la perte de dérivée par leur ordre. Si un grand nombre de travaux ont été déjà consacrés à la construction et à l'étude d'algèbres d'opérateurs à coefficients variables, y compris des travaux très intéressants sur le groupe de Heisenberg, notre approche est différente et en particulier elle conduit à la notion de direction microlocale, et complète l'élaboration d'une analyse microlocale sur le groupe de Heisenberg commencée par Bahouri, Gérard et Xu en 2000 par le développement d'une théorie de Littlewood-Paley.

CONTENTS

CHAPTER 1

INTRODUCTION AND MAIN RESULTS

1.1. Introduction

1.1.1. The Heisenberg group. — The Heisenberg group is obtained by constructing the group of unitary operators on $L^2(\mathbb{R}^n)$ generated by the n-dimensional group of translations and the n-dimensional group of multiplications (see for instance the book by M. Taylor [**53**]). It is an unimodular, nilpotent Lie group whose Haar measure coincides with the Lebesgue measure, and its remarkable feature is that its representation theory is rich as well as simple in structure. It is actually the first locally compact group whose infinite-dimensional, irreducible representations were classified (see [**22**]). It can be identified with a subgroup of the group of $(n+2)\times(n+2)$ real matrices with 1's on the diagonal and 0's below the diagonal.

It has a dual nature, in the sense that it may be realized as the boundary of the unit ball in several complex variables (thus extending to several complex variables the role played by the upper half plane and the Hilbert transform on its boundary) as well as being closely tied to quantum theory (via the Heisenberg commutators). We refer to the book by E. Stein [**52**], Chapter XII, for a comprehensive presentation of that duality.

Harmonic analysis on the Heisenberg group is a subject of constant interest, due on the one hand to its rich structure (though simple compared to other noncommutative Lie groups), and on the other hand to its importance in various areas of mathematics, from Partial Differential Equations (see among others [**7**], [**12**], [**16**] [**29**], [**30**], [**44**], [**45**], [**59**], [**60**]) to Geometry (see [**2**], [**18**], [**31**], [**47**]) or Number Theory (see for instance [**42**], [**55**]). Many research articles and monographs have been devoted to harmonic analysis on the Heisenberg group, and we shall give plenty of references as we go along.

1.1.2. Microlocal analysis on $\mathbb{R}^n$. — Microlocal analysis in the euclidian space appeared in the early seventies ([**50**]-[**51**]), and has at its foundation the theory of pseudodifferential operators. The main idea of microlocal analysis is to study a function simultaneously in the space variables of the physical space and in the

Fourier variables. Indeed, some phenomenon need both analysis to be correctly understood. As an example, let us consider the obstuctions to the convergence to zero in $L^2(\mathbb{R}^d)$ of two sequences, one of the form $u_n = h_n^{-d/2}\phi\left(\frac{x-x_0}{h_n}\right)$ and the other of the form $v_n = \exp\left(i\frac{(x\cdot\xi_0)}{h_n}\right)\phi(x)$ where $h_n \to 0$ and ϕ is in the Schwartz class for example. Of course, the point x_0 is a point of *concentration* in the space variables for the sequence u_n and as such, a point of obstruction to strong convergence to zero of the sequence. Similarly the *oscillations* in the direction ξ_0 correspond to *concentration* in Fourier variables for the sequence v_n, and they are also an obstruction to the strong convergence of the sequence.

With this point of view, it appears crucial to be able to use localization operators in space variables *and* in frequencies: the latter are Fourier multipliers. The theory of pseudodifferential operators provides a framework in which both points of view are unified: multiplication operators *and* Fourier multipliers are indeed pseudodifferential operators. More precisely, a pseudodifferential operator is defined by its *symbol* which is a function on the phase space: the symbol of the operator of multiplication by $\phi(x)$ is the function $(x,\xi) \mapsto \phi(x)$ and the symbol of the Fourier multiplier $\chi(D)$ is the function $(x,\xi) \mapsto \chi(\xi)$.

With pseudodifferential operators comes the concept of properties which hold *microlocally.* A function f satisfies a property (P) locally if for all cut-off function χ, the function χf satisfies (P); similarly, replacing the functions χ by a pseudodifferential operator with symbol supported in a given subset Ω of the phase-space, one gets a property satisfied microlocally in Ω. This notion allows a closer perception of the singularities of a function: in the 70's was developed the notion of *wave fronts*, analytic wave front, $\mathscr{C}^\infty$ wave front, etc. The idea is to associate with a given function f a region of the phase space where, microlocally, f is analytic or $\mathscr{C}^\infty$ or whatever else: this region is by definition the complement of the wave front.

One should notice that the phase space corresponds to the space of positions-impulsions of Quantum Mechanics, and thus enjoys nice geometric properties. It can be understood as the cotangent space to $\mathbb{R}^d$ (or to a submanifold if one works on a manifold) and is a symplectic space once endowed with the adapted symplectic form. This geometric aspect has been used successfully in numerous works and is one of the satisfying aspects of microlocal analysis (see for example the development of microlocal defect measures, semi-classical measures and Wigner measures as in **[34]** and **[35]** for example).

Microlocal analysis allowed for a very general study and classification of linear Partial Differential Equations with variable coefficients, using for example Littlewood-Paley operators which select a range of frequencies; such operators are pseudodifferential operators. In the case of nonlinear Partial Differential Equations, the situation is of course much more complicated, but paradifferential calculus (**[13]**) turned out to be a very powerful tool, for instance to analyze the propagation of singularities of solutions to such equations, or to study the associate Cauchy problem (see for instance **[3]**, in the case of quasilinear wave equations).