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**BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM
IN ARBITRARY LIPSCHITZ DOMAINS**

Marius MITREA & Matthew WRIGHT

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Marius MITREA & Matthew WRIGHT

Abstract. — The goal of this work is to treat the main boundary value problems for the Stokes system, i.e.,

- (i) the Dirichlet problem with L^p -data and nontangential maximal function estimates,
 - (ii) the Neumann problem with L^p -data and nontangential maximal function estimates,
 - (iii) the Regularity problem with L_1^p -data and nontangential maximal function estimates,
 - (iv) the transmission problem with L^p -data and nontangential maximal function estimates,
 - (v) the Poisson problem with Dirichlet condition in Besov-Triebel-Lizorkin spaces,
 - (vi) the Poisson problem with Neumann condition in Besov-Triebel-Lizorkin spaces,
- in Lipschitz domains of arbitrary topology in \mathbb{R}^n , for each $n \geq 2$. Our approach relies on boundary integral methods and yields constructive solutions to the aforementioned problems.

Résumé (Problèmes au bord pour le système de Stokes dans les domaines de Lipschitz quelconques.) — Le but de ce travail est d'étudier des problèmes au bord pour le système de Stokes, i.e.,

- (i) le problème de Dirichlet avec des données L^p et des estimations de la fonction maximale non tangentielle,
- (ii) le problème de Neumann avec des données L^p et des estimations de la fonction maximale non tangentielle,
- (iii) le problème de régularité avec des données L_1^p et des estimations de la fonction maximale non tangentielle,
- (iv) le problème de transmission avec des données L^p et des estimations de la fonction maximale non tangentielle,
- (v) le problème de Poisson avec des conditions de Dirichlet au bord dans des espaces de Besov-Triebel-Lizorkin,

(vi) le problème de Poisson avec des conditions de Neumann au bord dans des espaces de Besov-Triebel-Lizorkin, dans des domaines lipschitziens de \mathbb{R}^n pour tout $n \geq 2$ de topologie arbitraire. Notre approche repose sur des méthodes d'intégrales au bord et fournit des solutions constructives aux problèmes ci-dessus.

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CHAPTER 1

INTRODUCTION

1.1. Description of main well-posedness results

Informally speaking, the goal of the present work is to prove optimal well-posedness results for (homogeneous and inhomogeneous) boundary-value problems for the Stokes system in Lipschitz domains with arbitrary topology, in all space dimensions and for all major types of boundary conditions (Dirichlet, Neumann, transmission). The boundary data is selected from Lebesgue, Sobolev, Hardy, Besov and Triebel-Lizorkin spaces and the smoothness of the solutions is measured accordingly.

At the core of our analysis is the transmission problem for the Stokes system, on which we wish to elaborate first. Let Ω be a Lipschitz domain in \mathbb{R}^n , $n \geq 2$, and define $\Omega_+ := \Omega$ and $\Omega_- = \mathbb{R}^n \setminus \bar{\Omega}$. The transmission boundary value problem for the Stokes system studied here is of the type

$$(1.1) \quad (T_\mu) \quad \begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm & \text{in } \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 & \text{in } \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g} \in L_1^p(\partial\Omega), \\ \partial_\nu^\lambda(\vec{u}_+, \pi_+) - \mu \partial_\nu^\lambda(\vec{u}_-, \pi_-) = \vec{f} \in L^p(\partial\Omega), \\ M(\nabla \vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega). \end{cases}$$

Here, Δ is the Laplacian, $\mu \in [0, 1)$ is a fixed parameter, and $\nu := \nu_+$ is the outward unit normal to Ω_+ . For $1 < p < \infty$, $L_1^p(\partial\Omega)$ is the classical L^p -based Sobolev spaces of order one on $\partial\Omega$, M denotes the non-tangential maximal operator (cf. (2.5)), and

$$(1.2) \quad \partial_\nu^\lambda(\vec{u}_\pm, \pi_\pm) := (\nabla \vec{u}_\pm^\top + \lambda \nabla \vec{u}_\pm) \vec{\nu} - \pi_\pm \vec{\nu}$$

is a family of co-normal derivatives, indexed by a parameter $\lambda \in \mathbb{R}$ (more detailed definitions are given in subsequent chapters). In this way, we can simultaneously treat various types of Neumann boundary conditions. For example, when $\lambda = 0$, (1.2) corresponds to the co-normal derivative treated in [32], whereas when $\lambda = 1$, (1.2) corresponds to the “slip condition” considered in [21].

Two closely related boundary value problems are the Neumann problem and the Dirichlet problem with (maximally) regular data:

$$(1.3) \quad (N) \begin{cases} \Delta \vec{u} = \nabla \pi & \text{in } \Omega, \\ \operatorname{div} \vec{u} = 0 & \text{in } \Omega, \\ \partial_\nu^\lambda(\vec{u}, \pi) = \vec{f} \in L^p(\partial\Omega), \\ M(\nabla \vec{u}), M(\pi) \in L^p(\partial\Omega) \end{cases} \quad (R) \begin{cases} \Delta \vec{u} = \nabla \pi & \text{in } \Omega, \\ \operatorname{div} \vec{u} = 0 & \text{in } \Omega, \\ \vec{u}|_{\partial\Omega} = \vec{g} \in L_1^p(\partial\Omega), \\ M(\nabla \vec{u}), M(\pi) \in L^p(\partial\Omega). \end{cases}$$

From this point forth, we will refer to (R) as the Regularity problem. Fabes, Kenig, and Verchota proved in [32] that (N) and (R) are well-posed if $2 - \varepsilon < p < 2 + \varepsilon$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$. Building on the work in [19], [69], Z. Shen has established in [77] a weak maximum principle for the Dirichlet problem for the Stokes system in Lipschitz domains in \mathbb{R}^3 . Interpolating this L^∞ bound with the L^p -estimates from [32] with p near 2 shows that the Dirichlet problem for the Stokes system in three-dimensional Lipschitz domains with data in L^p is solvable whenever $2 - \varepsilon < p < \infty$. However, as pointed out by P. Dering on p. 16 of [28], “*this leaves open the question of whether these solutions may be constructed by means of the boundary layer method, and how to deal with exterior problems and slip boundary conditions.*”

With these aims in mind, let us briefly discuss the relevance of the transmission problem itself. From a physical point of view, the transmission problem

$$(1.4) \quad (T) \begin{cases} \mu_\pm \Delta \vec{u}_\pm = \nabla \pi_\pm & \text{in } \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 & \text{in } \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g}, \\ \sigma^\lambda \vec{u}_+ - \sigma^\lambda \vec{u}_- = \vec{f}, \end{cases}$$

where

$$(1.5) \quad \sigma^\lambda \vec{u}_\pm := \mu_\pm (\nabla \vec{u}_\pm^\top + \lambda \nabla \vec{u}_\pm) \vec{\nu} - \pi_\pm \vec{\nu},$$

describes the flow of a viscous incompressible fluid within and around a stationary particle occupying the domain Ω_+ which is further embedded into a second porous medium Ω_- . In this context, \vec{u}_+ and π_+ are the volume-averaged fluid velocity and pressure fields of the inner flow, whereas \vec{u}_- and π_- have analogous roles for the outer flow. In the specific case when $\lambda = 1$, this is a standard problem that arises when studying the low Reynolds number deformation of a viscous drop immersed in another fluid (see [73]; [71], Sec. 7.2). Here, μ_+ denotes the viscosity of the drop, while μ_- denotes the viscosity of the surrounding fluid. The case when $\vec{g} = 0$ is often of particular interest, since this introduces the physically relevant restriction that the velocities \vec{u}_+ and \vec{u}_- must match on the boundary. The reader is referred to M. Kohr and I. Pop’s monograph [51] for a more detailed discussion in this regard and for ample references to the engineering literature dealing with transmission problems for the Stokes system.

If we re-denote the term $\mu_\pm \vec{u}_\pm$ in (1.4) as simply \vec{u}_\pm and let $\mu := \mu_- / \mu_+$ denote the ratio of the viscosities of the two fluids, we can rewrite the transmission problem