344

ASTÉRISQUE

2012

BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM IN ARBITRARY LIPSCHITZ DOMAINS

Marius MITREA & Matthew WRIGHT

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque est un périodique de la Société mathématique de France.

Numéro 344, 2012

Comité de rédaction

Ahmed Abbes, Damien Gaboriau
Viviane Baladi, Michael Harris
Gérard Besson, Fabrice Planchon
Laurent Berger, Pierre Schapira
Philippe Biane, Bertrand Toen
Hélène Esnault
Éric Vasserot (dir.)

Diffusion

Maison de la SMF	Hindustan Book Agency	AMS
Case 916 - Luminy	O-131, The Shopping Mall	P.O. Box 6248
13288 Marseille Cedex 9	Arjun Marg, DLF Phase 1	Providence RI 02940
France	Gurgaon 122002, Haryana	USA
smf@smf.univ-mrs.fr	Inde	www.ams.org

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement Europe : 472 €, hors Europe : 512 € ($ 768)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

© *Société Mathématique de France 2012*

Tous droits réservés (article L 122–4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335–2 et suivants du CPI.

ISSN 0303-1179

ISBN 978-2-85629-343-0

Directeur de la publication : Aline BONAMI

344

ASTÉRISQUE

2012

BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM IN ARBITRARY LIPSCHITZ DOMAINS

Marius MITREA & Matthew WRIGHT

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Marius Mitrea
Department of Mathematics
University of Missouri at Columbia
Columbia, MO 65211, USA
mitream@missouri.edu

Matthew Wright
Department of Mathematics
Missouri State University
901 S. National Ave.
Springfield, MO 65897, USA
mwright@missouristate.edu

***Classification mathématique par sujet* (2000).** — 35J25, 42B20, 46E35; 35J05, 45B05, 31B10.

Mots-clefs. — Système de Stokes, domaine de Lipschitz, problèmes au bord, potentiels de couche, espaces de Bésov-Triebel-Lizorkin.

BOUNDARY VALUE PROBLEMS FOR THE STOKES SYSTEM IN ARBITRARY LIPSCHITZ DOMAINS

Marius MITREA & Matthew WRIGHT

Abstract. — The goal of this work is to treat the main boundary value problems for the Stokes system, i.e.,

(i) the Dirichlet problem with L^p-data and nontangential maximal function estimates,
(ii) the Neumann problem with L^p-data and nontangential maximal function estimates,
(iii) the Regularity problem with L^p_1-data and nontangential maximal function estimates,
(iv) the transmission problem with L^p-data and nontangential maximal function estimates,
(v) the Poisson problem with Dirichlet condition in Besov-Triebel-Lizorkin spaces,
(vi) the Poisson problem with Neumann condition in Besov-Triebel-Lizorkin spaces,

in Lipschitz domains of arbitrary topology in $\mathbb{R}^n$, for each $n \geq 2$. Our approach relies on boundary integral methods and yields constructive solutions to the aforementioned problems.

Résumé **(Problèmes au bord pour le système de Stokes dans les domaines de Lipschitz quelconques.)** — Le but de ce travail est d'étudier des problèmes au bord pour le système de Stokes, i.e.,

(i) le problème de Dirichlet avec des données L^p et des estimations de la fonction maximale non tangentielle,
(ii) le problème de Neumann avec des données L^p et des estimations de la fonction maximale non tangentielle,
(iii) le problème de régularité avec des données L^p_1 et des estimations de la fonction maximale non tangentielle,
(iv) le problème de transmission avec des données L^p et des estimations de la fonction maximale non tangentielle,
(v) le problème de Poisson avec des conditions de Dirichlet au bord dans des espaces de Besov-Triebel-Lizorkin,

© Astérisque 344, SMF 2012

(vi) le problème de Poisson avec des conditions de Neumann au bord dans des espaces de Besov-Triebel-Lizorkin,

dans des domaines lipschitziens de $\mathbb{R}^n$ pour tout $n \geq 2$ de topologie arbitraire. Notre approche repose sur des méthodes d'intégrales au bord et fournit des solutions constructives aux problèmes ci-dessus.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Description of main well-posedness results

Informally speaking, the goal of the present work is to prove optimal well-posedness results for (homogeneous and inhomogeneous) boundary-value problems for the Stokes system in Lipschitz domains with arbitrary topology, in all space dimensions and for all major types of boundary conditions (Dirichlet, Neumann, transmission). The boundary data is selected from Lebesgue, Sobolev, Hardy, Besov and Triebel-Lizorkin spaces and the smoothness of the solutions is measured accordingly.

At the core of our analysis is the transmission problem for the Stokes system, on which we wish to elaborate first. Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and define $\Omega_+ := \Omega$ and $\Omega_- = \mathbb{R}^n \setminus \bar{\Omega}$. The transmission boundary value problem for the Stokes system studied here is of the type

$$(1.1) \qquad (T_\mu) \quad \begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm \ \text{ in } \ \Omega_\pm, \\ \operatorname{div} \vec{u}_\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \vec{u}_+|_{\partial\Omega} - \vec{u}_-|_{\partial\Omega} = \vec{g} \in L^p_1(\partial\Omega), \\ \partial^\lambda_\nu(\vec{u}_+, \pi_+) - \mu\, \partial^\lambda_\nu(\vec{u}_-, \pi_-) = \vec{f} \in L^p(\partial\Omega), \\ M(\nabla \vec{u}_\pm), M(\pi_\pm) \in L^p(\partial\Omega). \end{cases}$$

Here, Δ is the Laplacian, $\mu \in [0, 1)$ is a fixed parameter, and $\nu := \nu_+$ is the outward unit normal to Ω_+. For $1 < p < \infty$, $L^p_1(\partial\Omega)$ is the classical L^p-based Sobolev spaces of order one on $\partial\Omega$, M denotes the non-tangential maximal operator (cf. (2.5)), and

$$(1.2) \qquad \partial^\lambda_\nu(\vec{u}_\pm, \pi_\pm) := (\nabla \vec{u}_\pm{}^\top + \lambda \nabla \vec{u}_\pm)\vec{\nu} - \pi_\pm \vec{\nu}$$

is a family of co-normal derivatives, indexed by a parameter $\lambda \in \mathbb{R}$ (more detailed definitions are given in subsequent chapters). In this way, we can simultaneously treat various types of Neumann boundary conditions. For example, when $\lambda = 0$, (1.2) corresponds to the co-normal derivative treated in [**32**], whereas when $\lambda = 1$, (1.2) corresponds to the "slip condition" considered in [**21**].

Two closely related boundary value problems are the Neumann problem and the Dirichlet problem with (maximally) regular data:

$$(1.3)\qquad (N)\begin{cases}\Delta\vec{u}=\nabla\pi \ \text{ in } \ \Omega,\\ \operatorname{div}\vec{u}=0 \ \text{ in } \ \Omega,\\ \partial_\nu^\lambda(\vec{u},\pi)=\vec{f}\in L^p(\partial\Omega),\\ M(\nabla\vec{u}),\, M(\pi)\in L^p(\partial\Omega)\end{cases}\qquad (R)\begin{cases}\Delta\vec{u}=\nabla\pi \ \text{ in } \ \Omega,\\ \operatorname{div}\vec{u}=0 \text{ in } \Omega,\\ \vec{u}|_{\partial\Omega}=\vec{g}\in L_1^p(\partial\Omega),\\ M(\nabla\vec{u}),\, M(\pi)\in L^p(\partial\Omega).\end{cases}$$

From this point forth, we will refer to (R) as the Regularity problem. Fabes, Kenig, and Verchota proved in [**32**] that (N) and (R) are well-posed if $2-\varepsilon<p<2+\varepsilon$, where $\varepsilon=\varepsilon(\partial\Omega)>0$. Building on the work in [**19**], [**69**], Z. Shen has established in [**77**] a weak maximum principle for the Dirichlet problem for the Stokes system in Lipschitz domains in $\mathbb{R}^3$. Interpolating this L^∞ bound with the L^p-estimates from [**32**] with p near 2 shows that the Dirichlet problem for the Stokes system in three-dimensional Lipschitz domains with data in L^p is solvable whenever $2-\varepsilon<p<\infty$. However, as pointed out by P. Deuring on p. 16 of [**28**], "*this leaves open the question of whether these solutions may be constructed by means of the boundary layer method, and how to deal with exterior problems and slip boundary conditions.*"

With these aims in mind, let us briefly discuss the relevance of the transmission problem itself. From a physical point of view, the transmission problem

$$(1.4)\qquad (T)\begin{cases}\mu_\pm\Delta\vec{u}_\pm=\nabla\pi_\pm \ \text{ in } \ \Omega_\pm,\\ \operatorname{div}\vec{u}_\pm=0 \ \text{ in } \ \Omega_\pm,\\ \vec{u}_+|_{\partial\Omega}-\vec{u}_-|_{\partial\Omega}=\vec{g},\\ \sigma^\lambda\vec{u}_+-\sigma^\lambda\vec{u}_-=\vec{f},\end{cases}$$

where

$$(1.5)\qquad \sigma^\lambda\vec{u}_\pm:=\mu_\pm(\nabla\vec{u}_\pm{}^\top+\lambda\nabla\vec{u}_\pm)\vec{\nu}-\pi_\pm\vec{\nu},$$

describes the flow of a viscous incompressible fluid within and around a stationary particle occupying the domain Ω_+ which is further embedded into a second porous medium Ω_-. In this context, $\vec{u}_+$ and π_+ are the volume-averaged fluid velocity and pressure fields of the inner flow, whereas $\vec{u}_-$ and π_- have analogous roles for the outer flow. In the specific case when $\lambda=1$, this is a standard problem that arises when studying the low Reynolds number deformation of a viscous drop immersed in another fluid (see [**73**]; [**71**], Sec. 7.2). Here, μ_+ denotes the viscosity of the drop, while μ_- denotes the viscosity of the surrounding fluid. The case when $\vec{g}=0$ is often of particular interest, since this introduces the physically relevant restriction that the velocities $\vec{u}_+$ and $\vec{u}_-$ must match on the boundary. The reader is referred to M. Kohr and I. Pop's monograph [**51**] for a more detailed discussion in this regard and for ample references to the engineering literature dealing with transmission problems for the Stokes system.

If we re-denote the term $\mu_\pm\vec{u}_\pm$ in (1.4) as simply $\vec{u}_\pm$ and let $\mu:=\mu_-/\mu_+$ denote the ratio of the viscosities of the two fluids, we can rewrite the transmission problem