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DEFORMATION QUANTIZATION MODULES

Masaki KASHIWARA & Pierre SCHAPIRA

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Abstract. — On a complex manifold $(X, \mathscr{O}_X)$, a DQ-algebroid $\mathscr{A}_X$ is an algebroid stack locally equivalent to the sheaf $\mathscr{O}_X[[\hbar]]$ endowed with a star-product and a DQ-module is an object of the derived category $\mathrm{D^b}(\mathscr{A}_X)$.

The main results are:

— the notion of cohomologically complete DQ-modules which allows one to deduce various properties of such a module $\mathscr{M}$ from the corresponding properties of the $\mathscr{O}_X$-module $\mathbb{Z}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X[\hbar]} \mathscr{M}$,

— a finiteness theorem, which asserts that the convolution of two coherent DQ-kernels defined on manifolds $X_i \times X_j$ $(i = 1, 2, j = i + 1)$, satisfying a suitable properness assumption, is coherent (a non commutative Grauert's theorem),

— the construction of the dualizing complex for coherent DQ-modules and a duality theorem which asserts that duality commutes with convolution (a non commutative Serre's theorem),

— the construction of the Hochschild class of coherent DQ-modules and the theorem which asserts that Hochschild class commutes with convolution,

— in the commutative case, the link between Hochschild classes and Chern and Euler classes,

— in the symplectic case, the constructibility (and perversity) of the complex of solutions of an holonomic DQ-module into another one after localizing with respect to $\hbar$.

Hence, these Notes could be considered both as an introduction to non commutative complex analytic geometry and to the study of microdifferential systems on complex Poisson manifolds.

***Résumé* (Modules de déformation quantification).** — Sur une variété complexe $(X, \mathscr{O}_X)$, un DQ-algebroide $\mathscr{A}_X$ est un champ d'algébroides localement équivalent au faisceau $\mathscr{O}_X[[\hbar]]$ muni d'un star-produit et un DQ-module est un objet de la catégorie dérivée $\mathrm{D^b}(\mathscr{A}_X)$.

Les résultats principaux sont :

— la notion de DQ-module cohomologiquement complet qui permet de déduire diverses propriétés d'un tel module $\mathscr{M}$ des propriétés correspondantes du $\mathscr{O}_X$-module $\mathbb{Z}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X[\hbar]} \mathscr{M}$,
— un théorème de finitude qui assure que la convolution de deux DQ-noyaux cohérents définis sur des variétés $X_i \times X_j$ $(i = 1, 2, j = i + 1)$, vérifiant certaines hypothèses de propreté, est cohérent (un théorème de Grauert non commutatif),
— la construction du complexe dualisant pour les DQ-modules cohérents et un théorème de dualité qui assure que la dualité commute avec la convolution (un théorème de Serre non commutatif),
— la construction de la classe de Hochschild des DQ-modules cohérents et le théorème qui assure que la classe de Hochschild commute avec la convolution,
— dans le cas commutatif, le lien entre classes de Hochschild et classes de Chern et de Euler,
— dans le cas symplectique, la constructibilité (et la perversité) du complexe des solutions d'un DQ-module holonome dans un autre, après localisation en $\hbar$.

Ces Notes peuvent donc être considérées à la fois comme une introduction à la géométrie analytique complexe non commutative et à l'étude des systèmes microdifférentiels sur les variétés de Poisson complexes.

CONTENTS

INTRODUCTION

In a few words these Notes could be considered both as an introduction to non commutative complex analytic geometry and to the study of microdifferential systems. Indeed, on a complex manifold X, we replace the structure sheaf $\mathscr{O}_X$ with a formal deformation of it, that is, a DQ-algebra, or better, a DQ-algebroid, and study modules over this ring, extending to this framework classical results of Cartan-Serre and Grauert, and also classical results on Hochschild classes and the index theorem. Here, DQ stands for "deformation quantization". But the theory of modules over DQ-algebroids is also a natural generalization of that of $\mathscr{D}$-modules. Indeed, when the Poisson structure underlying the deformation is symplectic, the study of DQ-modules naturally generalizes that of microdifferential modules, and sometimes makes it easier (see Theorem 7.2.3).

The notion of a star product is now a classical subject studied by many authors and naturally appearing in various contexts. Two cornerstones of its history are the paper **[2]** (see also **[4, 5]**) who defines $\star$-products and the fundamental result of **[48]** which, roughly speaking, asserts that any real Poisson manifold may be "quantized", that is, endowed with a star algebra to which the Poisson structure is associated. It is now a well-known fact (see **[37, 47]**) that, in order to quantize complex Poisson manifolds, sheaves of algebras are not well-suited and have to be replaced by algebroid stacks. We refer to **[16, 64]** for further developments.

In this paper, we consider complex manifolds endowed with DQ-algebroids, that is, algebroid stacks locally associated to sheaves of star-algebras, and study modules over such algebroids. The main results of this paper are:

— a finiteness theorem, which asserts that the convolution of two coherent kernels, satisfying a suitable properness assumption, is coherent (a kind of Grauert's theorem),
— the construction of the dualizing complex and a duality theorem, which asserts that duality commutes with convolution,
— the construction of the Hochschild class of coherent DQ-modules and the theorem which asserts that Hochschild class commutes with convolution,
— the link between Hochschild classes and Chern classes and also with Euler classes, in the commutative case,
— the constructibility of the complex of solutions of an holonomic module into another one in the symplectic case.

Let us describe this paper with some details.

In Chapter 1, we systematically study rings (*i.e.,* sheaves of rings) which are formal deformations of rings, and modules over such deformed rings. More precisely, consider a topological space X, a commutative unital ring $\mathbb{K}$ and a sheaf $\mathscr{A}$ of $\mathbb{K}[[\hbar]]$-algebras on X which is $\hbar$-complete and without $\hbar$-torsion. We also assume that there exists a base of open subsets of X, acyclic for coherent modules over $\mathscr{A}_0 := \mathscr{A}/\hbar\mathscr{A}$.

We first show how to deduce various properties of the ring $\mathscr{A}$ from the corresponding properties on $\mathscr{A}_0$. For example, $\mathscr{A}$ is a Noetherian ring as soon as $\mathscr{A}_0$ is a Noetherian ring, and an $\mathscr{A}$-module $\mathscr{M}$ is coherent as soon as it is locally finitely generated and $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is $\mathscr{A}_0$-coherent for all $n \geq 0$. Then, we introduce the property of being cohomologically complete for an object of the derived category $\mathrm{D}(\mathscr{A})$. We prove that this notion is local, stable by direct images and an object $\mathscr{M}$ with bounded coherent cohomology is cohomologically complete. Conversely, if $\mathscr{M}$ is cohomologically complete, it has coherent cohomology objects as soon as its graded module $\mathscr{A}_0 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}$ has coherent cohomology over $\mathscr{A}_0$ (see Theorem 1.6.4). We also give a similar criterion which ensures that an $\mathscr{A}$-module is flat.

In Chapter 2 we consider the case where X is a complex manifold, $\mathbb{K} = \mathbb{C}$, $\mathscr{A}_0 = \mathscr{O}_X$ and $\mathscr{A}$ is locally isomorphic to an algebra $(\mathscr{O}_X[[\hbar]], \star)$ where $\star$ is a star-product. It is an algebra over $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$. We call such an algebra $\mathscr{A}$ a DQ-algebra. We also consider DQ-algebroids, that is, $\mathbb{C}^\hbar$-algebroids (in the sense of stacks) locally equivalent to the algebroid associated with a DQ-algebra. Remark that a DQ-algebroid on a manifold X defines a Poisson structure on it. Conversely, a famous theorem of Kontsevich [**48**] asserts that on a real Poisson manifold there exists a DQ-algebra to which this Poisson structure is associated. In the complex case, there is a similar result using DQ-algebroids. This is a theorem of [**47**] after a related result of [**37**] in the contact case.

If $(X, \mathscr{A}_X)$ is a complex manifold X endowed with a DQ-algebroid $\mathscr{A}_X$, we denote by X^a the manifold X endowed with the DQ-algebroid $\mathscr{A}_X^{\mathrm{op}}$ opposite to $\mathscr{A}_X$.

We define the external product $\mathscr{A}_{X_1 \times X_2}$ of two DQ-algebroids $\mathscr{A}_{X_1}$ and $\mathscr{A}_{X_2}$ on manifolds X_1 and X_2. There exists a canonical $\mathscr{A}_{X \times X^a}$-module $\mathscr{C}_X$ on $X \times X^a$ supported by the diagonal, which corresponds to the $\mathscr{A}_X$-bimodule $\mathscr{A}_X$.

On a complex manifold X endowed with a DQ-algebroid, we construct the $\mathbb{C}^\hbar$-algebroid $\mathscr{D}_X^{\mathscr{A}}$, a deformation quantization of the ring $\mathscr{D}_X$ of differential operators. It is a $\mathbb{C}^\hbar$-subalgebroid of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$. It turns out that $\mathscr{D}_X^{\mathscr{A}}$ is equivalent to $\mathscr{D}_X[[\hbar]]$. This new algebroid allows us to construct the dualizing complex $\omega_X^{\mathscr{A}}$ associated to a DQ-algebroid $\mathscr{A}_X$. This complex is the dual over $\mathscr{D}_X^{\mathscr{A}}$ of $\mathscr{A}_X$, similarly to the case of $\mathscr{O}_X$-modules. Note that the dualizing complex for DQ-algebras has already been considered in a more particular situation by [**23, 24**].

We also adapt to algebroids a results of [**42**] which allows us to replace a coherent $\mathscr{A}_X$-module by a complex of "almost free" modules, such an object being a locally finite sum $\oplus_{i \in I}(L_i)_{U_i}$, the L_i's being free $\mathscr{A}_X$-modules of finite rank defined on a neighborhood of $\overline{U_i}$. We give a similar result for algebraic manifolds.