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Frédéric PAULIN, Mark POLLICOTT and Barbara SCHAPIRA

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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EQUILIBRIUM STATES IN NEGATIVE CURVATURE

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Abstract. — With their origin in thermodynamics and symbolic dynamics, Gibbs measures are crucial tools to study the ergodic theory of the geodesic flow on negatively curved manifolds. We develop a framework (through Patterson-Sullivan densities) allowing us to get rid of compactness assumptions on the manifold, and prove many existence, uniqueness and finiteness results of Gibbs measures. We give many applications, to the Variational Principle, the counting and equidistribution of orbit points and periods, the unique ergodicity of the strong unstable foliation and the classification of Gibbs densities on some Riemannian covers.

***Résumé* (États d'équilibre en courbure négative.)** — Les mesures de Gibbs, utilisées d'abord en thermodynamique et en dynamique symbolique, sont des outils cruciaux pour l'étude de la théorie ergodique des flots géodésiques des variétés de courbure strictement négative. Nous introduisons (via les densités de Patterson-Sullivan) un cadre qui permet de s'affranchir d'hypothèses de compacité sur la variété, et nous démontrons de nombreux résultats d'existence, d'unicité et de finitude des mesures de Gibbs. Nous en donnons moultes applications, au principe variationnel, au comptage et à l'équidistribution des points d'orbites et des périodes, à l'unique ergodicité du feuilletage fortement instable, et à la classification des mesures de Gibbs sur certains revêtements riemanniens.

CONTENTS

CHAPTER 1

INTRODUCTION

With their thermodynamic origin, Gibbs measures (or states) are very useful in symbolic dynamics over finite alphabets (see for instance [140], [158], [83]). Sinai, Bowen and Ruelle introduced them in hyperbolic dynamics (which, via coding theory, has strong links to symbolic dynamics), in particular in order to study the weighted distribution of periodic orbits and the equilibrium states for given potentials (see for instance [149], [25], [27], [119]). For instance, this allows the dynamical analysis of the geodesic flow of negatively curved Riemannian manifolds, provided its non-wandering set is compact. As a first step to venture beyond the compact case, Gibbs measures in symbolic dynamics over countable alphabets have been developed by Sarig [142], [144]. But since no coding theory which does not lose geometric information is known for general non-compact manifolds, this methodology is not well adapted to the non-compact case. Note that a coding-free approach of transfer operators, in particular via improved spectral methods, was put in place by Liverani, Baladi, Gouëzel and others (see for instance, restricting the references to the case of flows, [93], [34], [153], [154], [12], [60], as well as [5] for an example in a non locally homogeneous and non-compact situation). In this text, we construct and study geometrically Gibbs measures for the geodesic flow of negatively curved Riemannian manifolds, without compactness assumptions.

By considering the action on a universal Riemannian cover of its covering group, this study is strongly related to the (weighted) distribution of orbits of discrete groups of isometries of negatively curved simply connected Riemannian manifolds. After work of Huber and Selberg (in particular through his trace formula) in constant curvature, Margulis (see for instance [99]) made a breakthrough, albeit in the unweighted lattice case, to give the precise asymptotic growth of the orbits. Patterson [121] and Sullivan [151], in the surface and constant curvature case respectively, have introduced a nice approach using measures at infinity, giving in particular a nice construction of the measure of maximal entropy in the cocompact case. The Patterson-Sullivan theory extends to variable curvature and non lattice case, and an optimal reference (in the unweighted case) is due to Roblin [134].

After preliminary work of Hamenstädt [66] on Gibbs cocycles, of Ledrappier [91], Coudène [38] and Schapira [147] using a slightly different approach, and of Mohsen [103] (their contributions will be explained as the story unfolds), the aim of this book is to develop an optimal theory of Gibbs measures for the dynamical study of geodesic flows in general negatively curved Riemannian manifolds, and of weighted distribution of orbits of general discrete groups of isometries of negatively curved simply connected Riemannian manifolds.

Let us give a glimpse of our results, starting by describing the players. Let M be a complete connected Riemannian manifold with pinched negative curvature. For simplicity in this introduction, we assume that the derivatives of the sectional curvature of M are uniformly bounded. Let $F : T^1M \to \mathbb{R}$ be a Hölder-continuous map, called a *potential*, which is going to help us define the various weights. In order to simplify the exposition of this introduction, we assume here that F is invariant by the antipodal map $v \mapsto -v$ (see the main body of the text for complete statements). Let $p : \widetilde{M} \to M$ be a universal Riemannian covering map, with covering group Γ and sphere at infinity $\partial_\infty \widetilde{M}$, and let $\widetilde{F} = F \circ p$. We make no compactness assumption on M: we only assume Γ to be non-elementary (that is, non virtually nilpotent). (In the body of the text, we will also allow M to be a good orbifold, hence Γ to have torsion.) Let $\phi = (\phi_t)_{t\in\mathbb{R}}$ be the geodesic flow on T^1M and $\widetilde{\phi} = (\widetilde{\phi}_t)_{t\in\mathbb{R}}$ the one on $T^1\widetilde{M}$. For all $x, y \in \widetilde{M}$, let us define

$$\int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}(\widetilde{\phi}_t v)\,\mathrm{d}t$$

where v is the unit tangent vector at x to a geodesic from x through y. For every periodic orbit g of ϕ, let $\mathscr{L}_g$ be the Lebesgue measure along g and $\int_g F = \mathscr{L}_g(F)$ the *period* of g for the potential F.

We will study three numerical invariants of the weighted dynamics.

• Let $x, y \in \widetilde{M}$, and $c > 0$ large enough. The *critical exponent* of (Γ, F) is an exponential growth rate of the orbit points of Γ weighted by the potential F:

$$\delta_{\Gamma,F} = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{\substack{\gamma\in\Gamma \\ t-c\le d(x,\gamma y)\le t}} \mathrm{e}^{\int_x^{\gamma y} \widetilde{F}}.$$

• For every $t \ge 0$, let $\mathscr{P}\mathrm{er}(t)$ be the set of periodic orbits of ϕ with length at most t, and $\mathscr{P}\mathrm{er}'(t)$ its subset of primitive ones. Let W be a relatively compact open subset of T^1M meeting the non-wandering set of ϕ, and $c > 0$ large enough. The *Gurevich pressure* of (M, F) is an exponential growth rate of the closed geodesics of M weighted by the potential F:

$$P_{\mathrm{Gur}}(M, F) = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{\substack{g\in\mathscr{P}\mathrm{er}(t)-\mathscr{P}\mathrm{er}(t-c) \\ g\cap W\neq\varnothing}} \mathrm{e}^{\int_g \widetilde{F}}.$$

The restriction for the periodic orbits under consideration to meet a given compact set is necessary: For instance if M is an infinite cyclic cover of a compact manifold,