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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Abstract. — Our goal in this volume is to apply a normal forms method to estimate the Sobolev norms of the solutions of the water waves equation. We construct a paradifferential change of unknown, without derivatives losses, which eliminates the part of the quadratic terms that bring non zero contributions in a Sobolev energy inequality. Our approach is purely Eulerian: we work on the Craig-Sulem-Zakharov formulation of the water waves equation.

In addition to these Sobolev estimates, we also prove L^2-estimates for the $\partial_x^\alpha Z^\beta$-derivatives of the solutions of the water waves equation, where Z is the Klainerman vector field $t\partial_t + 2x\partial_x$. These estimates are used in the paper [**6**]. In that reference, we prove a global existence result for the water waves equation with smooth, small, and decaying at infinity Cauchy data, and we obtain an asymptotic description in physical coordinates of the solution, which shows that modified scattering holds. The proof of this global in time existence result relies on the simultaneous bootstrap of some Hölder and Sobolev a priori estimates for the action of iterated Klainerman vector fields on the solutions of the water waves equation. The present volume contains the proof of the Sobolev part of that bootstrap.

***Résumé* (Estimations Sobolev pour les ondes de gravité en dimension deux.)** — Le but de ce volume est d'appliquer une méthode de formes normales à l'estimation des normes Sobolev des solutions de l'équation des ondes de surface avec gravité. Nous éliminons, par un changement de variables paradifférentiel sans perte de dérivées, les termes quadratiques de la nonlinéarité qui pourraient générer une croissance de l'inégalité d'énergie Sobolev. Notre approche est purement Eulérienne, basée sur la formulation de Craig-Sulem-Zakharov des ondes de gravité.

Outre ces estimations Sobolev, nous prouvons également des bornes L^2 pour l'action de $\partial_x^\alpha Z^\beta$ sur les solutions de l'équation, Z désignant le champ de Klainerman $t\partial_x + 2x\partial_t$. Ces inégalités sont utilisées dans l'article [**6**]. Dans cette référence, nous prouvons un résultat d'existence globale pour l'équation des ondes de gravité à données de Cauchy petites, régulières, décroissantes à l'infini, et nous obtenons une description asymptotique dans l'espace physique de la solution, qui montre qu'il y

a diffusion modifiée. La démonstration de ce résultat d'existence globale repose sur une méthode d'induction simultanée, pour des estimations a priori portant sur des normes Hölder ou Sobolev de la solution de l'équation, sur laquelle agissent des itérés de champs de Klainerman. Le présent volume est consacré à l'obtention du volet Sobolev de ces estimations.

CONTENTS

INTRODUCTION

1. Description of the main results

This paper addresses the well-posedness of the initial value problem for the motion of a two-dimensional incompressible fluid under the influence of gravity. At time t, the fluid domain, denoted by $\Omega(t)$, has a free boundary described by the equation $y=\eta(t,x)$, so that

$$\Omega(t)=\left\{\,(x,y)\in\mathbb{R}^2\,;\,y<\eta(t,x)\,\right\}.$$

The velocity field $v\colon\Omega\to\mathbb{R}^2$ is assumed to be irrotational and to satisfy the incompressible Euler equations. It follows that $v=\nabla_{x,y}\phi$ for some velocity potential $\phi\colon\Omega\to\mathbb{R}$ satisfying

$$\Delta_{x,y}\phi=0,\quad \partial_t\phi+\frac{1}{2}\left|\nabla_{x,y}\phi\right|^2+P+gy=0, \tag{1.1}$$

where $g>0$ is the acceleration of gravity, P is the pressure term, $\nabla_{x,y}=(\partial_x,\partial_y)$ and $\Delta_{x,y}=\partial_x^2+\partial_y^2$. Hereafter, the units of length and time are chosen so that $g=1$.

The water waves equations are then given by two boundary conditions on the free surface:

$$\begin{cases}\partial_t\eta=\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi & \text{on } \partial\Omega,\\ P=0 & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where ∂_n is the outward normal derivative of Ω, so that $\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi=\partial_y\phi-(\partial_x\eta)\partial_x\phi$.

It is well known that the linearized equation around the equilibrium $\eta=0$ and $\phi=0$ can be written under the form $\partial_t^2u+|D_x|\,u=0$ where $|D_x|$ is the Fourier multiplier with symbol $|\xi|$. Allowing oneself to oversimplify the problem, one can think of the linearized equation around a nontrivial solution as the equation $(\partial_t+V\partial_x)^2u+a\,|D_x|\,u=0$, where V is the trace of the horizontal component of the velocity at the free surface and $a=-\partial_yP|_{y=\eta}$ is the so-called Taylor coefficient. To insure that the Cauchy problem for the latter equation is well-posed, one has to require that a is bounded from below by a positive constant. This is known as the Taylor sign condition; see [**41**] for an ill-posedness result without this requirement. That the well-posedness of the Cauchy problem depends on an assumption on the sub-principal term $a\,|D_x|$ reflects the fact that the linearized equation has a double characteristic, see Craig [**29**, Section 4] or Lannes [**56**, Section 4.1]. This leads to an apparent loss of 1/2 derivative in the study of the Cauchy problem in Sobolev spaces. However, Nalimov [**68**] proved that, in Lagrangian coordinates, the Cauchy problem is well-posed locally in time, in the framework of Sobolev spaces, under an additional smallness assumption on the data; see also the results of Yosihara [**87**] and Craig [**28**].

Notice that if η and ϕ are of size ε then $a = 1 + O(\varepsilon)$ so that the Taylor sign condition is satisfied for ε small enough. As was first proved by Wu [**83, 84**], this property is always true, without smallness assumption (including the case that the interface is not a graph, as long as the interface is non self-intersecting). As a result, the well-posedness of the Cauchy problem was proved in [**83, 84**] without smallness assumption. Several extensions or different proofs are known and we refer the reader to Córdoba, Córdoba and Gancedo [**25**], Coutand-Shkoller [**26**], Lannes [**56, 59, 60**], Linblad [**62**], Masmoudi-Rousset [**64**], Shatah-Zeng [**73, 74**], Zhang-Zhang [**89**] for recent results concerning the gravity water waves equations.

Two different approaches were used in the analysis of the water waves equations: the Lagrangean formulation with a more geometrical point of view and the Eulerian formulation in relation with microlocal analysis. Our analysis is entirely based on the Eulerian formulation of the water waves equations: we shall work on the so-called Craig–Sulem–Zakharov system which we introduce below. Let us also mention that the idea of studying the water waves equations by means of microlocal analysis is influenced by the papers by Craig-Schanz-Sulem [**33**], Lannes [**56**] and Iooss-Plotnikov [**53**]. More precisely, we follow the paradifferential analysis introduced in [**7**] and further developed in [**1, 5**]. We explain later in this introduction how this allows to overcome the apparent loss of derivative in the Cauchy problem.

Following Zakharov [**88**] and Craig and Sulem [**34**], we work with the trace of ϕ at the free boundary

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

and introduce the Dirichlet-Neumann operator $G(\eta)$ that relates ψ to the normal derivative $\partial_n\phi$ of the potential by

$$(G(\eta)\psi)(t,x) = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta(t,x)}.$$

Then (η,ψ) solves (see [**34**]) the system

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + \eta + \dfrac{1}{2}(\partial_x\psi)^2 - \dfrac{1}{2(1+(\partial_x\eta)^2)}\big(G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)\big)^2 = 0. \end{cases} \tag{1.3}$$

Consider a classical solution (η,ψ) of (1.3), such that (η,ψ) belongs to $C^0([0,T];H^s(\mathbb{R}))$ for some $T > 0$ and $s > 3/2$. Then it is proved in [**3**] that there exist a velocity potential ϕ and a pressure P satisfying (1.1) and (1.2). Thus it is sufficient to solve the Craig–Sulem–Zakharov formulation (1.3) of the water waves equations (1.1)-(1.2).

Our goal in this paper is to apply a normal forms method to estimate the Sobolev norms of the solutions to the water waves equations. In practice, one looks for a