Fabrice Castel

GEOMETRIC REPRESENTATIONS OF THE BRAID GROUPS

ASTÉRISQUE 378

Société Mathématique de France 2016

Astérisque est un périodique de la Société mathématique de France.
Numéro 378

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs 2016

Vente au numéro : 45 € ($ 67)
Abonnement Europe : 654 €, hors Europe : 692 € ($ 1038)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0303-1179
ISBN 978-2-85629-835-0

Directeur de la publication : Marc Peigné

ASTÉRISQUE 378

GEOMETRIC REPRESENTATIONS OF THE BRAID GROUPS

Fabrice Castel

Société Mathématique de France 2016
Publié avec le concours du Centre National de la Recherche Scientifique

F. Castel
Institut de Mathématiques de Bourgogne, UMR 5584 CNRS,
Université de Bourgogne, Faculté des Sciences Mirande, 9 avenue Alain Savary,
BP 47870, 21078 Dijon Cedex (France).
E-mail : `fabrice.castel@gmail.com`

2010 *Mathematics Subject Classification*. — Primary 20F38, 57M07. Secondary 57M99, 20F36, 20E36, 57M05.

***Key words and phrases*. —** surface, mapping class group, braid group, rigidity, geometric representation, Nielsen Thurston's classification, transvection, monodromy morphism.

Research partially supported by National University of Singapore Research Grant R-146-000-137-112.

GEOMETRIC REPRESENTATIONS OF THE BRAID GROUPS

Fabrice Castel

Abstract. — We define a *geometric representation* to be any representation of a group in the mapping class group of a surface. Let $\Sigma_{g,b}$ be the orientable connected compact surface of genus g with b boundary components, and $\mathcal{PM}\mathrm{od}(\Sigma_{g,\,b})$ the associated mapping class group globally preserving each boundary component. The aim of this paper consists in describing the set of the geometric representations of the braid group $\mathcal{B}_n$ with $n \geqslant 6$ strands in $\mathcal{PM}\mathrm{od}(\Sigma_{g,\,b})$ subject to the only condition that $g \leqslant n/2$. We prove that under this condition, such representations are either *cyclic*, that is, their images are cyclic groups, or are *transvections of monodromy homomorphisms*, that is, up to multiplication by an element in the centralizer of the image, the image of a standard generator of $\mathcal{B}_n$ is a Dehn twist, and the images of two consecutive standard generators are two Dehn twists along two curves intersecting in one point.

This leads to different results. They will be proved in later papers, but we explain how they are deduced from our main theorem. These corollaries concern four families of groups: the braid groups $\mathcal{B}_n$ for all $n \geqslant 6$, the Artin groups of type D_n for all $n \geqslant 6$, the mapping class groups $\mathcal{PM}\mathrm{od}(\Sigma_{g,\,b})$ (preserving each boundary component) and the mapping class groups $\mathcal{M}\mathrm{od}(\Sigma_{g,\,b},\, \partial\Sigma_{g,\,b})$ (preserving the boundary pointwise), for $g \geqslant 2$ and $b \geqslant 0$.

We describe the remarkable structure of the sets of the endomorphisms of these groups, their automorphisms and their outer automorphism groups. We also describe the set of the homomorphisms between braid groups $\mathcal{B}_n \to \mathcal{B}_m$ with $m \leqslant n+1$ and the set of the homomorphisms between mapping class groups of surfaces (possibly with boundary) whose genera (greater than or equal to 2) differ by at most one. Finally, we describe the set of the geometric representations of the Artin groups of type E_n ($n \in \{6,\, 7,\, 8\}$).

***Résumé* (Représentations géométriques des groupes de tresses)**
On appelle *représentation géométrique* toute représentation d'un groupe dans le groupe de difféotopies, couramment appelé "mapping class group", d'une surface. Soient $\Sigma_{g,b}$ la surface orientable, compacte et connexe de genre g possédant b composantes de bord et $\mathcal{PM}\mathrm{od}(\Sigma_{g,b})$ le groupe de difféotopies associé préservant chaque composante de bord. Le but de cet article est de décrire l'ensemble des représentations géométriques du groupe de tresses $\mathcal{B}_n$ à $n \geqslant 6$ brins dans $\mathcal{PM}\mathrm{od}(\Sigma_{g,b})$, pour tous n, g, b pourvu que $g \leqslant n/2$. On prouve que sous cette seule condition, de telles représentations sont soit *cycliques*, c'est-à-dire d'image cyclique, soit *des transvections d'homomorphismes de monodromie*, c'est-à-dire qu'à multiplication près par un élément du centralisateur de l'image, l'image d'un générateur standard de $\mathcal{B}_n$ est un twist de Dehn, et l'image de deux générateurs standards consécutifs sont deux twists de Dehn le long de deux courbes s'intersectant en un point.

De nombreux corollaires en découlent. On les prouvera dans de futurs articles, mais on explique ici comment chacun se déduit de notre théorème principal. Ces corollaires concernent cinq familles de groupes : les groupes de tresses $\mathcal{B}_n$ pour tout $n \geqslant 6$, les groupes d'Artin de type D_n pour tout $n \geqslant 6$, les groupes d'Artin de type E_n pour tout $n \in \{6, 7, 8\}$, les groupes de difféotopies $\mathcal{PM}\mathrm{od}(\Sigma_{g,b})$ (préservant chaque composante de bord) et les groupes de difféotopies $\mathcal{M}\mathrm{od}(\Sigma_{g,b}, \partial\Sigma_{g,b})$ (préservant le bord point par point), pour tout $g \geqslant 2$ et $b \geqslant 0$. Pour chacune de ces familles, excepté les groupes de type E_n, on décrira précisément la structure (toujours remarquable) des endomorphismes, on déterminera les endomorphismes injectifs, les automorphismes et le groupe d'automorphismes extérieurs. On décrira également l'ensemble des homomorphismes entre groupes de tresses $\mathcal{B}_n \to \mathcal{B}_m$ avec $m \leqslant n+1$ et l'ensemble des homomorphismes entre groupes de difféotopies de surfaces (éventuellement à bords) dont les genres (supérieurs à 2) diffèrent d'au plus 1. Concernant les groupes d'Artin de type E_n, on décrira toutes leurs représentations géométriques, et l'on déduira d'un théorème de Waynryb qu'aucune n'est injective.

CONTENTS

CHAPTER 0

INTRODUCTION

Some theorems have been established concerning injective endomorphisms of the braid groups or the mapping class groups. The outer automorphisms groups were first computed, in 1981 by Dyer and Grossman, [**17**], for the braid groups, and in 1986 − 1988 by Ivanov, [**25**], and McCarthy, [**40**], for the mapping class groups. Later, the set of the injective endomorphisms was described, in 1999 by Ivanov and McCarthy, [**28**], for the mapping class groups, and in 2006 by Bell and Margalit, [**2**], for the braid groups. For both families of groups, the previous authors could say a little more on injective endomorphisms between two different braid groups or the mapping class groups of two distinct surfaces, with strong restrictions, though. Our main theorem concerns the homomorphisms from the braid group in the mapping class group, making the bridge between both families of groups. As corollaries, we are able to prove the previous theorems and even to strengthen them by getting rid of the injective hypothesis.

0.1. Presentation of the main objects

The surfaces in this paper will always be compact, orientable and oriented 2-manifold with each connected component having negative Euler characteristic. They may have nonempty boundary. Classically, we denote by $\partial\Sigma$ the topological boundary of a surface Σ, we denote by $\Sigma_{g,b}$ the connected surface of genus g with b boundary components and we denote by $\chi(\Sigma)$ the Euler characteristic of a surface Σ, for example: $\chi(\Sigma_{g,b}) = 2 - 2g - b \leqslant -1$. Implicitly, the integers g and b will always satisfy $2 - 2g - b \leqslant -1$.

The mapping class groups $\mathcal{M}\mathrm{od}(\Sigma)$ of a surface Σ is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ. Thus, boundary Dehn twists (see below) are trivial in $\mathcal{M}\mathrm{od}(\Sigma)$. We denote by $\mathcal{PM}\mathrm{od}(\Sigma)$ the finite index subgroup of $\mathcal{M}\mathrm{od}(\Sigma)$

consisting of elements of $\mathcal{M}\mathrm{od}(\Sigma)$ that permute neither the connected components, nor the boundary components.

The relative mapping class groups $\mathcal{M}\mathrm{od}(\Sigma, \partial\Sigma)$ is the group of isotopy classes of all orientation-preserving diffeomorphisms that coincide with the identity on $\partial\Sigma$. Thus, boundary Dehn twists (see below) are not trivial in $\mathcal{M}\mathrm{od}(\Sigma)$.

A geometric representation of a group G is an homomorphism from G to the mapping class group of some surface $\Sigma_{g,b}$. For instance, J. Birman, A. Lubotsky and J. McCarthy have shown in [**6**] that the maximal rank of the abelian subgroups of the mapping class group $\mathcal{M}\mathrm{od}(\Sigma_{g,b})$ is equal to the complexity $\xi(\Sigma_{g,b}) = 3g - 3 + b$ of $\Sigma_{g,b}$. In other words, they have shown that the group $\mathbb{Z}^n$ has faithful geometric representations in $\mathcal{M}\mathrm{od}(\Sigma_{g,b})$ if and only if $n \leqslant 3g - 3 + b$ holds. In this paper, we will investigate geometric representations of the braid group.

The braid group $\mathcal{B}_n$ on n strands is the group defined by the following presentation, which we will call the *classic presentation*:

$$\mathcal{B}_n = \left\langle \tau_1, \tau_2, \ldots, \tau_{n-1} \;\middle|\; \begin{cases} \tau_i\tau_j = \tau_j\tau_i & \text{if } |i-j| \neq 1 \\ \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j & \text{if } |i-j| = 1 \end{cases} \right\rangle.$$

The generators of this presentation are called the *standard generators of* $\mathcal{B}_n$. Our conventions are the same as in [**29**]: the braids are drawn from top to bottom; when applying τ_i the $i+1$-st strand passes over the i-th strand as in Figure 0.1.1; by projection on the punctures disk, we get a left half-twist; at last, a product of two braids $\alpha\beta$ consists in the braid α (above) followed by the braid β (below).

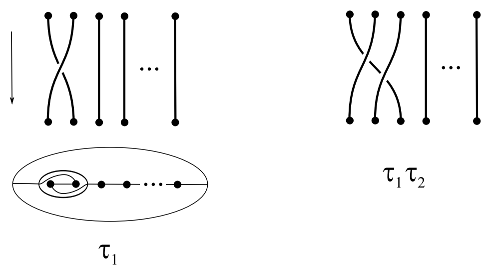

FIGURE 0.1.1. The first standard generator τ_1; the product $\tau_1\tau_2$.

0.2. The main theorems: Theorems 1, 2 and 3

The aim of this paper is to describe all geometric representations of the braid group $\mathcal{B}_n$ in the mapping class groups $\mathcal{PM}\mathrm{od}(\Sigma_{g,b})$ and $\mathcal{M}\mathrm{od}(\Sigma_{g,b}, \partial\Sigma_{g,b})$. The only