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***Abstract*.** --- We study a variant of the Alt, Caffarelli, and Friedman free boundary problem, with many phases and a slightly different volume term, which we originally designed to guess the localization of eigenfunctions of a Schrödinger operator in a domain. We prove Lipschitz bounds for the functions and some nondegeneracy and regularity properties for the domains.

***Résumé* (Un problème de frontière libre pour la localisation de fonctions propres)**
On étudie une variante du problème de frontière libre de Alt, Caffarelli et Friedman, avec plusieurs phases et un terme de volume légèrement différent, que l'on a choisie pour deviner la localisation des fonctions propres d'un opérateur de Schrödinger dans un domaine. On démontre des estimations lipschitziennes pour les fonctions associées à un minimiseur et des propriétés de nondégérescence et de régularité pour les frontières libres.

CONTENTS

CHAPTER 1

INTRODUCTION

The initial motivation for this paper was to describe the localization of eigenfunctions for an operator $\mathscr{L}$ on a domain $\Omega \subset \mathbb{R}^n$. Let us assume that $|\Omega|$, the measure of Ω, is finite. The typical operator that we consider is the positive Laplacian $\mathscr{L} = -\Delta$, or a Schrödinger operator $\mathscr{L} = -\Delta + \mathscr{V}$, with $\mathscr{V}$ bounded and nonnegative.

In [**20**], a pointwise estimate for eigenfunctions for $\mathscr{L}$ is found, which bounds them in terms of a single function w_0, namely the solution of $\mathscr{L} w_0 = 1$ on Ω, with the Dirichlet condition $w_0 = 0$ on $\mathbb{R}^n \setminus \Omega$. Our first goal is to derive an automatic way, using w_0, to find subdomains W_j, $1 \leq j \leq N$, of Ω, where the eigenfunctions of $\mathscr{L}$ are more likely to be supported. The work in [**20**] indicates that, roughly speaking, one seeks a collection of disjoint $W_j \subset \Omega$, $1 \leq j \leq N$, such that w_0 is small on the boundaries of the W_j, and it is natural to try to measure "smallness" in terms of the operator $\mathscr{L}$ itself. Even though many handmade or numerical decompositions of Ω based on w_0 seem to give very good predictions of the localization of eigenfunctions, we would like to have a more systematic way to realize the decomposition.

The functional described below is intended to give such a good partition of Ω into subdomains, and it turns out to be an interesting variant of functionals introduced by Alt and Caffarelli [**1**], and studied by many others. In the present paper, we shall mainly study the theoretical properties of our functional (existence and regularity of the minimizers and regularity of the corresponding free boundaries).

Let us now describe the main free boundary problem that we shall study here; the relation with w_0 and our original localization problem will be explained more in Chapter 2.

We are given a domain $\Omega \subset \mathbb{R}^n$, and (for instance) an operator $\mathscr{L} = -\Delta + \mathscr{V}$; assumptions on the potential $\mathscr{V}$, or other functions associated to a similar problem, will come later. We are also given an integer $N \geq 1$, and we want to cut Ω into subregions W_i, $1 \leq i \leq N$, according to the geometry associated to $\mathscr{L}$. For this, we want to define and minimize a functional J. But let us first define the set of admissible pairs $(\boldsymbol{u}, W)$ for which $J(\boldsymbol{u}, W)$ is defined.

Definition 1.1. — Given the open set $\Omega \subset \mathbb{R}^n$ and the integer $N \geq 1$, we denote by $\mathcal{F} = \mathcal{F}(\Omega)$ the set of admissible pairs $(\boldsymbol{u}, \boldsymbol{W})$, where $\boldsymbol{W} = (W_i)_{1\leq i\leq N}$ is a N-uple of pairwise disjoint Borel-measurable sets $W_i \subset \Omega$, and $\boldsymbol{u} = (u_i)_{1\leq i\leq N}$ is a N-uple of real-valued functions u_i, such that

$$u_i \in W^{1,2}(\mathbb{R}^n) \tag{1.1}$$

and

$$u_i(x) = 0 \text{ for almost every } x \in \mathbb{R}^n \setminus W_i. \tag{1.2}$$

Here $W^{1,p}(\mathbb{R}^n)$, $1 \leq p < +\infty$, denotes the set of functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ whose derivative, computed in the distribution sense, lies in $L^p(\mathbb{R}^n)$. We chose a definition for which we do not need to assume any regularity for the sets W_i, nor to give a precise meaning to the Sobolev space $W^{1,2}_0(W_i)$, but under mild assumptions, the functions u_i associated to minimizers will be continuous, and we will be able to take

$$W_i = \{x \in \Omega\,;\ u_i(x) > 0\}.$$

For the moment we took real-valued functions, but what we will say will systematically apply when some of the functions u_i are required to be nonnegative. In addition, some of our results will only work under this constraint (that $u_i \geq 0$).

Our functional J will have three main terms. The first one is the energy

$$E(\boldsymbol{u}) = \sum_{i=1}^{N} \int \left|\nabla u_i(x)\right|^2 \mathrm{d}x, \tag{1.3}$$

where we denoted by ∇u_i the distributional gradient of u_i, which is an L^2 function. It does not matter whether we integrate on $\mathbb{R}^n$, Ω, or W_i, because one can check that if $u_i \in W^{1,2}(\mathbb{R}^n)$ vanishes almost everywhere on $\mathbb{R}^n \setminus W_i$, then $\nabla u_i = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$. Indeed, the Rademacher-Calderón theorem says that u_i is differentiable (i.e., has a derivative) at almost every point of $\mathbb{R}^n$, with a differential that coincides almost everywhere with the (density of the) distribution Du_i. See for instance Theorem 6.15 on page 47 of [**23**]. It is then easy to check that $Du_i(x) = 0$ when x is a point of Lebesgue differentiability of $\mathbb{R}^n \setminus W_i$, hence, for almost every $\mathbb{R}^n \setminus W_i$.

The second term of our functional will be

$$M(\boldsymbol{u}) = \sum_{i=1}^{N} \int \left[u_i(x)^2 f_i(x) - u_i(x) g_i(x)\right] \mathrm{d}x, \tag{1.4}$$

where the f_i and the g_i, $1 \leq i \leq N$, are given functions on Ω that we may choose, depending on our problem. We are slightly abusing notation here, because $M(\boldsymbol{u})$ also depends on the W_i through the choice of functions we integrate against u_i^2 or u_i, at least if the f_i and g_i depend on i. The convergence of the integrals in (1.4) will follow from our assumptions on the f_i and the g_i. For our initial localization problem, all the f_i will be equal to the potential $\mathcal{V}$, and $g_i = 2$ for all i.