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THE REALIZATION SPACE OF AN UNSTABLE COALGEBRA

by Georg BIEDERMANN, Georgios RAPTIS & Manfred STELZER

Abstract. — Unstable coalgebras over the Steenrod algebra form a natural target category for singular homology with prime field coefficients. The realization problem asks whether an unstable coalgebra is isomorphic to the homology of a topological space. We study the moduli space of such realizations and give a description of this in terms of cohomological invariants of the unstable coalgebra. This is accomplished by a thorough comparative study of the homotopy theories of cosimplicial unstable coalgebras and of cosimplicial spaces.

***Résumé* (Espaces de réalisation d'une coalgèbre instable).** — L'homologie singulière à coefficients dans un corps premier d'un espace topologique a la structure d'une coalgèbre instable sur l'algèbre de Steenrod. La question de savoir si une coalgèbre instable donnée est isomorphe à l'homologie d'un espace topologique est le problème de réalisation. Nous décrivons une tour d'espaces qui converge vers l'espace de module des réalisations sous des conditions raisonables. La différence entre deux niveaux consécutifs est controllée par la cohomologie des coalgèbres instables. Ces résultats sont obtenus par une étude comparative approfondie des théories d'homotopie des coalgèbre instables cosimpliciales et des espaces cosimpliciaux.

CONTENTS

CHAPTER 1

INTRODUCTION

Let p be a prime and C an unstable (co)algebra over the Steenrod algebra $\mathcal{A}_p$. The realization problem asks whether C is isomorphic to the singular $\mathbb{F}_p$-(co)homology of a topological space X. In case the answer turns out to be affirmative, one may further ask how many such spaces X there are up to $\mathbb{F}_p$-homology equivalence. The purpose of this work is to study a moduli space of topological realizations associated with a given unstable coalgebra C. Our results give a description of this moduli space in terms of a tower of spaces which are determined by cohomological invariants of C. As a consequence, we obtain obstruction theories for the existence and uniqueness of topological realizations, where the obstructions are defined by André-Quillen cohomology classes. These obstruction theories recover and sharpen results of Blanc [**7**]. Moreover, our results apply also to the case of rational coefficients. We thus provide a unified picture of the theory in positive and zero characteristics.

Let us start by briefly putting the problem into historical perspective. The realization problem was explicitly posed by Steenrod in [**58**]. In the rational context, at least if one restricts to simply-connected objects, such realizations always exist by celebrated theorems of Quillen [**48**] and Sullivan [**59**]. In contrast, there are many deep non-realization theorems known in positive characteristic: some of the most notable ones are by Adams [**2**], Liulevicius [**40**], Ravenel [**50**], Hill-Hopkins-Ravenel [**36**], Schwartz [**55**], and Gaudens-Schwartz [**28**].

Moduli spaces parametrizing homotopy types with a given cohomology algebra or homotopy Lie algebra were first constructed in rational homotopy theory. The case of cohomology was treated by Félix [**27**], Lemaire-Sigrist [**39**], and Schlessinger-Stasheff [**53**]. All these works relied on the obstruction theory developed by Halperin-Stasheff [**33**]. The (moduli) set of equivalence classes of realizations was identified with the quotient of a rational variety by the action of a unipotent algebraic group. Moreover, Schlessinger and Stasheff [**53**] associated to a graded algebra A a differential graded coalgebra C_A, whose set of components, defined in terms of an algebraic notion of homotopy, parametrizes the different realizations of A. This coalgebra represents a moduli space for A which encodes higher order information.

An obstruction theory for unstable coalgebras was developed by Blanc in [7]. He defined obstruction classes for realizing an unstable coalgebra C and proved that the vanishing of these classes is necessary and sufficient for the existence of a realization. Furthermore, he defined difference classes which distinguish two given realizations. Both kinds of classes live in certain André-Quillen cohomology groups associated to C. Earlier, Bousfield [12] developed an obstruction theory for realizing maps with obstruction classes in the unstable Adams spectral sequence. For very nice unstable algebras, obstruction theories using the Massey-Peterson machinery [42] were also introduced by Harper [34] and McCleary [43] already in the late seventies.

In a landmark paper, Blanc, Dwyer, and Goerss [8] studied the moduli space of topological realizations of a given Π-algebra over the integers. Again, the components of this moduli space correspond to different realizations. By means of simplicial resolutions, the authors showed a decomposition of this moduli space into a tower of fibrations, thus obtaining obstruction theories for the realization and uniqueness problems for Π-algebras. Their work relied on earlier work of Dwyer, Kan and Stover [24] on resolution (or E^2) model categories which was later generalized by Bousfield [13]. Using analogous methods, Goerss and Hopkins in [29], and [30], studied the moduli space of E_∞-algebras in spectra which have a prescribed homology with respect to a homology theory. Their results gave rise to some profound applications in stable homotopy theory [52].

In this book, we consider an unstable coalgebra C over the Steenrod algebra and we study the (possibly empty) moduli space of realizations $\mathcal{M}_{\mathrm{Top}}(C)$. We work with unstable coalgebras and homology, instead of unstable algebras and cohomology, because one avoids in this way all kinds of issues that are related to (non-)finiteness. In the presence of suitable finiteness assumptions, the two viewpoints can be translated into each other. We exhibit a decomposition of $\mathcal{M}_{\mathrm{Top}}(C)$ into a tower of fibrations in which the layers have homotopy groups related to the André-Quillen cohomology groups of C. Our results apply also to the rational case. The decomposition of $\mathcal{M}_{\mathrm{Top}}(C)$ is achieved by means of cosimplicial resolutions and their Postnikov decompositions in the cosimplicial direction, following and dualizing the approach of [8]. At several steps, the application of this general approach to moduli space problems from [8] requires non-trivial input and this accounts in part for the length of the article. Another reason is that the article is essentially self-contained.

We will now give the precise statements of our main results. Following the work of Dwyer and Kan on moduli spaces in homotopy theory, a moduli space of objects in a certain homotopy theory is defined as the classifying space of a category of weak equivalences. Under favorable homotopical assumptions, it turns out that this space encodes the homotopical information of the spaces of homotopy automorphisms of these objects. We refer to Appendix C for the necessary background and a detailed list of references.