

**Yiannis Sakellaridis**  
**Akshay Venkatesh**

---

**PERIODS AND HARMONIC ANALYSIS  
ON SPHERICAL VARIETIES**

---

ASTÉRISQUE 396

Société Mathématique de France 2017

---

Astérisque est un périodique de la Société mathématique de France  
Numéro 396

---

Comité de rédaction

Ahmed ABBES	Philippe EYSSIDIEUX
Viviane BALADI	Damien GABORIAU
Laurent BERGER	Michael HARRIS
Philippe BIANE	Fabrice PLANCHON
Hélène ESNAULT	Pierre SCHAPIRA

Éric VASSEROT (dir.)

Diffusion

Maison de la SMF B.P. 67 13274 Marseille Cedex 9 France christian.smf@cirm-math.fr	AMS P.O. Box 6248 Providence RI 02940 USA www.ams.org
--	---

Tarifs 2017

*Vente au numéro* : 60 € (\$ 90)  
*Abonnement électronique* : 500 € (\$ 750)  
*Abonnement avec supplément papier* : 657 €, hors Europe : 699 € (\$ 1049)  
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque

Société Mathématique de France

Institut Henri Poincaré, 11, rue Pierre et Marie Curie  
75231 Paris Cedex 05, France  
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96  
astsmf@ihp.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2017

*Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.*

ISSN 0303-1179 (print) 2492-5926 (electronic)

ISBN 978-2-85629-871-8

Stéphane SEURET  
Directeur de la publication

---

ASTÉRIQUE 396

**PERIODS AND HARMONIC ANALYSIS  
ON SPHERICAL VARIETIES**

**Yiannis Sakellaridis  
Akshay Venkatesh**

**Société Mathématique de France 2017**  
Publié avec le concours du Centre National de la Recherche Scientifique

*Y. Sakellaridis*

*E-mail* : sakellar@rutgers.edu

Department of Mathematics and Computer Science, Rutgers University,  
Newark, NJ, USA.

*A. Venkatesh*

*E-mail* : akshay@math.stanford.edu

Department of Mathematics, Stanford University, Stanford, CA, USA.

---

**2010 Mathematics Subject Classification.** — 22E50, 11F70.

**Key words and phrases.** — Spherical varieties, Plancherel formula, relative Langlands program, periods of automorphic forms.

---

# PERIODS AND HARMONIC ANALYSIS ON SPHERICAL VARIETIES

Yiannis Sakellaridis, Akshay Venkatesh

**Abstract.** — Given a spherical variety  $X$  for a group  $G$  over a non-Archimedean local field  $k$ , the Plancherel decomposition for  $L^2(X)$  should be related to “distinguished” Arthur parameters into a dual group closely related to that defined by Gaitsgory and Nadler. Motivated by this, we develop, under some assumptions on the spherical variety, a Plancherel formula for  $L^2(X)$  up to discrete (modulo center) spectra of its “boundary degenerations”, certain  $G$ -varieties with more symmetries which model  $X$  at infinity. Along the way, we discuss the asymptotic theory of subrepresentations of  $C^\infty(X)$  and establish conjectures of Ichino-Ikeda and Lapid-Mao. We finally discuss global analogues of our local conjectures, concerning the period integrals of automorphic forms over spherical subgroups.

**Résumé.** — Ce volume développe l'idée selon laquelle l'analyse harmonique d'une variété sphérique  $X$  est étroitement liée au programme de Langlands. Dans le cas local, la conjecture principale dit que la décomposition spectrale de  $L^2(X)$  est contrôlée par un groupe dual attaché à  $X$ . En poursuivant cette idée, les auteurs établissent une formule de Plancherel pour  $L^2(X)$ , faisant intervenir des variétés sphériques plus simples qui apparaissent dans la géométrie du bord de  $X$ . Cette étude locale est ensuite reliée aux conjectures globales sur les périodes de formes automorphes le long de sous-groupes sphériques.



# CONTENTS

<b>1. Introduction</b> .....	1
<b>Part I. The dual group of a spherical variety</b> .....	19
<b>2. Review of spherical varieties</b> .....	21
2.1. Invariants .....	22
2.2. The dual group of a spherical variety .....	25
2.3. Toroidal compactifications .....	29
2.4. Normal bundles and boundary degenerations .....	33
2.5. Degeneration to the normal bundle; affine degeneration .....	37
2.6. Whittaker-type induction .....	41
2.7. Levi varieties .....	47
2.8. Horocycle space .....	48
2.9. The example of $\mathbf{PGL}(V)$ as a $\mathbf{PGL}(V) \times \mathbf{PGL}(V)$ variety .....	50
<b>3. Proofs of the results on the dual group</b> .....	55
3.1. The root datum of a spherical variety .....	55
3.2. Distinguished morphisms .....	59
3.3. The work of Gaitsgory and Nadler .....	60
3.4. Uniqueness of a distinguished morphism .....	63
3.5. The identification of the dual group .....	65
3.6. Commuting $\mathrm{SL}_2$ .....	67
<b>Part II. Local theory and the Ichino-Ikeda conjecture</b> .....	69
<b>4. Geometry over a local field</b> .....	71
4.1. Measures .....	71
4.2. The measure on $X_\Theta$ .....	72
4.3. Exponential map .....	74

<b>5. Asymptotics</b> .....	81
5.1. The main result .....	81
5.2. Proof of asymptotics .....	86
5.3. Cartan decomposition and matrix coefficients .....	93
5.4. Mackey theory, the Radon transform and asymptotics .....	98
5.5. Support of elements in $e_{\Theta}^*(C_c^{\infty}(X))$ .....	103
<b>6. Strongly tempered varieties</b> .....	109
6.1. Abstract Plancherel decomposition .....	109
6.2. Definition; the canonical hermitian form .....	112
6.3. The Whittaker case and the Lapid-Mao conjecture .....	115
6.4. The Ichino-Ikeda conjecture .....	123
<b>Part III. Spectral decomposition and scattering theory</b> .....	131
<b>7. Results</b> .....	133
7.1. Plancherel decomposition and direct integrals of Hilbert spaces .....	134
7.2. Discrete spectrum .....	135
7.3. Main result .....	136
<b>8. Two toy models: the global picture and semi-infinite matrices</b> .....	139
8.1. Global picture .....	139
8.2. Spectra of semi-infinite matrices: scattering theory on $\mathbb{N}$ .....	141
<b>9. The discrete spectrum</b> .....	149
9.1. Decomposition according to the center .....	149
9.2. A finiteness result .....	151
9.3. Variation with the central character .....	155
9.4. Toric families of relative discrete series .....	159
9.5. Unfolding .....	166
<b>10. Preliminaries to the Bernstein morphisms: “linear algebra”</b> .....	177
10.1. Basic definitions .....	177
10.2. Finite and polynomial functions .....	178
10.3. Hermitian forms .....	182
10.4. Measurability of eigenprojections .....	185
<b>11. The Bernstein morphisms</b> .....	189
11.1. Main result .....	189
11.2. Harish-Chandra-Schwartz space and temperedness of exponents ..	191
11.3. Plancherel formula for $X_{\Theta}$ from Plancherel formula for $X$ .....	195
11.4. The Bernstein maps. Equivalence with Theorem 11.1.2 .....	199
11.5. Property characterizing the Bernstein maps .....	201
11.6. Compatibility with composition and inductive structure of $L^2(X)$ ..	203
11.7. Isometry .....	204



<b>12. Preliminaries to scattering (I): direct integrals and norms</b> .....	207
12.1. General properties of the Plancherel decomposition .....	207
12.2. Norms on direct integrals of Hilbert spaces .....	210
<b>13. Preliminaries to scattering (II): consequences of the conjecture on discrete series</b> .....	219
13.1. Isogenies of tori and affine maps on their character groups .....	219
13.2. Relationship between central characters for $X_\Theta$ and $X_\Omega$ .....	223
13.3. Canonical decomposition of maps $L^2(X_\Theta) \rightarrow L^2(X_\Omega)$ .....	226
<b>14. Scattering theory</b> .....	231
14.1. Introduction .....	231
14.2. Generic injectivity of the map $\mathfrak{a}_X^*/W_X \rightarrow \mathfrak{a}^*/W$ .....	232
14.3. The scattering theorem .....	234
14.4. Proof of the first part of Theorem 14.3.1 .....	240
14.5. Estimates .....	245
14.6. Proof of the second part of Theorem 14.3.1 .....	251
<b>15. Explicit Plancherel formula</b> .....	255
15.1. Goals .....	255
15.2. Various spaces of coinvariants .....	258
15.3. Convergence issues and affine embeddings .....	263
15.4. Normalized Eisenstein integrals and smooth asymptotics .....	273
15.5. The canonical quotient and the small Mackey restriction .....	278
15.6. Unitary asymptotics (Bernstein maps) .....	281
15.7. The group case .....	286
<b>Part IV. Conjectures</b> .....	291
<b>16. The local <math>X</math>-distinguished spectrum</b> .....	293
16.1. Recollection of the Arthur conjectures .....	293
16.2. The conjecture on the local spectrum (weak form) .....	296
16.3. A global to local argument .....	298
16.4. Proof of Theorem 16.3.2 .....	302
16.5. Pure inner forms .....	305
<b>17. Speculation on a global period formula</b> .....	311
17.1. Tamagawa measure .....	312
17.2. Factorization and the Ichino-Ikeda conjecture .....	312
17.3. Local prerequisites for the conjecture .....	314
17.4. Global conjecture .....	317
17.5. How to understand the Euler product .....	318
17.6. Everywhere discrete or unramified .....	322

<b>18. Examples</b> .....	325
18.1. Principal Eisenstein periods .....	325
18.2. Parabolic periods .....	331
18.3. The Whittaker case for $GL_n$ .....	334
18.4. Compatibility of the conjecture with unfolding .....	339
<b>A. Prime rank one spherical varieties</b> .....	345
A.1. Goals .....	345
A.2. Lie algebra versions .....	346
A.3. Reductions for the existence statement .....	346
A.4. Reductions for the uniqueness statement .....	348
A.5. Further discussion of the existence result .....	350
A.6. Proof of Lemma A.4.2 .....	351
<b>Bibliography</b> .....	353