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Ciprian Manolescu

**THE EQUIVALENCE OF TWO SEIBERG-WITTEN
FLOER HOMOLOGIES**

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Abstract. — We show that monopole Floer homology (as defined by Kronheimer and Mrowka) is isomorphic to the S^1 -equivariant homology of the Seiberg-Witten Floer spectrum constructed by the second author.

Résumé (Équivalence des deux homologie de Seiberg-Witten Floer)

Dans ce volume, nous montrons que l'homologie de Floer des monopoles (telle que définie par Kronheimer et Mrowka) est isomorphe à l'homologie S^1 -équivariante du spectre de Seiberg-Witten Floer construit par le second auteur.

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CHAPTER 1

INTRODUCTION

1.1. BACKGROUND

The Seiberg-Witten (monopole) equations [45], [46] are an important tool for understanding the topology of smooth four-dimensional manifolds. A signed count of the solutions of these equations on a closed four-manifold yields the Seiberg-Witten invariant [51]. On a four-manifold with boundary, instead of a numerical invariant one can define an element in a group associated to the boundary, called the Seiberg-Witten Floer homology. There are several different constructions of Seiberg-Witten Floer homology in the literature, [32], [28], [18], [13]. The goal of this monograph is to prove that, for rational homology spheres, the definitions given by Kronheimer-Mrowka in [18] and by the second author in [28] are equivalent.

The construction in [18] applies to an arbitrary three-manifold Y , equipped with a Spin^c structure s . Given this data, Kronheimer and Mrowka define an infinite dimensional analog of the Morse complex, with the underlying space being the blow-up of the configuration space of Spin^c connections and spinors (modulo gauge). The role of gradient flow lines is played by solutions to generic perturbations of the Seiberg-Witten equations on $\mathbb{R} \times Y$. Their invariant, monopole Floer homology, is the homology of the resulting complex. This complex (and hence also its homology) comes with a $\mathbb{Z}[U]$ -module structure. The applications of monopole Floer homology include the surgery characterization of the unknot [19] and Taubes' proof of the Weinstein conjecture in three dimensions [48].

In fact, there are three different versions of monopole Floer homology defined in [18]; they are denoted \widetilde{HM} , \widehat{HM} , and \overline{HM} . Yet another version, \widetilde{HM} , was constructed by Bloom in [3]: To define \widetilde{HM} , one considers the cone of the U map on the complex that defines \widehat{HM} , and then takes homology.

Compared with [18], the construction in [28] was originally done only for rational homology spheres; on the other hand, it yields something more than a homology group. By using finite dimensional approximation of the Seiberg-Witten equations, combined

with Conley index theory, one obtains an invariant in the form of an equivariant suspension spectrum. Specifically, given a rational homology sphere Y equipped with a Spin^c structure \mathfrak{s} , one can associate to it an S^1 -equivariant spectrum $\text{SWF}(Y, \mathfrak{s})$. (See also [17], [16], [43] for extensions of this construction to the case $b_1 > 0$.)

The S^1 -equivariant homology of $\text{SWF}(Y, \mathfrak{s})$ can be viewed as a definition of Seiberg-Witten Floer homology. The advantage of having a Floer spectrum is that one can also apply other (equivariant) generalized homology functors to it. For example, by adding the conjugation symmetry, one can define a $\text{Pin}(2)$ -equivariant Seiberg-Witten Floer homology; this was instrumental in the disproof of the triangulation conjecture by the second author [31]. For other applications of the Floer spectrum, see [29], [30], [27] and [26].

1.2. RESULTS

The bulk of this monograph is devoted to proving:

THEOREM 1.1. — *Let Y be a rational homology sphere with a Spin^c structure \mathfrak{s} . There is an isomorphism of absolutely-graded $\mathbb{Z}[U]$ -modules:*

$$\widetilde{HM}_*(Y, \mathfrak{s}) \cong \widetilde{H}_*^{S^1}(\text{SWF}(Y, \mathfrak{s})),$$

where \widetilde{HM} is the “to” version of monopole Floer homology defined in [18], and $\widetilde{H}_*^{S^1}$ denotes reduced equivariant (Borel) homology.¹

From Theorem 1.1 we deduce that Bloom’s homology \widetilde{HM} can be identified with the ordinary (non-equivariant) homology of the Floer spectrum SWF :

COROLLARY 1.2. — *Let Y be a rational homology sphere equipped with a Spin^c structure \mathfrak{s} . Then, $\widetilde{HM}_*(Y, \mathfrak{s}) \cong \widetilde{H}_*(\text{SWF}(Y, \mathfrak{s}))$ as absolutely graded abelian groups.*

From the absolute grading on monopole Floer homology one can extract a \mathbb{Q} -valued invariant, called the Frøyshov invariant; see [13] or [18, Section 39.1]. A similar numerical invariant, called δ , was defined in [31, Section 3.7] using the Floer spectrum SWF . (The definition there was only given for Spin structures, but it extends to the Spin^c setting.) An immediate consequence of Theorem 1.1 is

COROLLARY 1.3. — *Let Y be a rational homology sphere equipped with a Spin^c structure \mathfrak{s} . Then, $\delta(Y, \mathfrak{s}) = -h(Y, \mathfrak{s})$, where h is the Frøyshov invariant as defined in [18, Section 39.1].*

1. In this book, we grade Borel homology so that we simply have $\widetilde{H}_*^{S^1}(X) = \widetilde{H}_*(X \wedge_{S^1} ES_+^1)$. This differs from the grading conventions in [14] or [17] by one.