Antoine Ducros

FAMILIES OF BERKOVICH SPACES

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***Abstract.* —** This book is devoted to the systematic study of relative properties in the context of Berkovich analytic spaces. We first develop a theory of flatness in this setting. After showing through a counter-example that naive flatness cannot be the right notion because it is not stable under base change, we define flatness by *requiring* invariance under base change, and we study a first important class of flat morphisms, that of quasi-smooth ones.

We then show the existence of local *dévissages* (in the spirit of Raynaud and Gruson's work) for coherent sheaves, which we use, together with a study of the local rings of "generic fibers" of morphisms, to prove that a *boundaryless*, naively flat morphism is flat.

After that we prove that the image of a compact strictly analytic space by a flat morphism is an analytic domain of the target and that it admits, when the source is stricly analytic, a compact, flat multisection (*i.e.*, a compact, flat cover of relative dimension zero over which there is a section). The image of a compact analytic space by a flat morphism had already been described by Raynaud in the rigid-analytic setting, but our method is completely different.

In the last part of this work we study where various interesting pointwise relative properties are satisfied. We first prove that the flat locus of a given morphism of analytic spaces is a Zariski-open subset of the source (we follow the method that was introduced by Kiehl for the complex analytic analogue of this statement). We then look at the loci at which a point satisfies various commutative algebra properties *on its fiber*: being geometrically regular, geometrically R_m, complete intersection, or Gorenstein; being S_m or Cohen-Macaulay. We prove that the results we could expect actually hold: these loci are (locally) Zariski-constructible, and Zariski-open under suitable extra assumptions (flatness, and also equidimensionality for S_m and geometric R_m); for that purpose, we first study the general properties of the locally Zariski-constructible subsets of an analytic space.

***Résumé* (*Familles d'espaces de Berkovich*). —** Ce livre est consacré à une étude systématique des propriétés relatives dans le contexte des espaces de Berkovich. Nous commençons par développer une théorie de la platitude dans ce cadre. L'acception naïve de cette notion est inadaptée : nous montrons en effet par un contre-exemple qu'elle n'est pas stable par changement de base, ce qui nous conduit à *imposer* cette stabilité dans la définition. Nous étudions une première classe importante de morphismes plats : celle des morphismes *quasi-lisses*.

Nous montrons ensuite l'existence de *dévissages* locaux (dans l'esprit de Raynaud et Gruson) pour les faisceaux cohérents. Joint à une étude des anneaux locaux des fibres « génériques » des morphismes, cela nous permet de montrer qu'un morphisme *sans bord* qui est plat au sens naïf l'est encore au nôtre.

Puis nous démontrons que l'image d'un espace strictement analytique compact par un morphisme plat est un domaine analytique du but, et qu'elle admet, lorsque la source est strictement analytique, une multisection compacte et plate (*i.e.* un revêtement plat, compact, et de dimension relative nulle sur lequel le morphisme considéré possède une section). L'image d'un espace analytique compact par un morphisme plat avait déjà été décrite par Raynaud dans le contexte rigide-analytique, mais notre méthode est complètement différente.

Dans la dernière partie de ce travail, nous étudions les lieux de validité sur la source de certaines propriétés relatives. Nous y démontrons pour commencer que le lieu de platitude d'un morphisme d'espaces analytiques est un ouvert de Zariski de la source (nous suivons la méthode utilisée par Kiehl pour établir l'assertion correspondante en géométrie analytique complexe). Cela nous permet de procéder à l'investigation systématique des ensemble de points satisfaisant *dans leur fibre* les propriétés classiques de l'algèbre commutative : être géométriquement régulier, géométriquement R_m, d'intersection complète ou de Gorenstein ; être S_n ou de Cohen-Macaulay. Nous prouvons que les énoncés auxquels ont peut s'attendre sont effectivement vérifiés : ces lieux de validités sont (localement) Zariski-constructibles, et sont des ouverts de Zariski sous certaines hypothèses supplémentaires (platitude, ainsi qu'équidimensionalité pour les propriétés R_m ou S_m) ; dans ce but, nous procédons tout d'abord à une étude générale des parties localement Zariski-constructibles d'un espace analytique.

CONTENTS

INTRODUCTION

This book is, roughly speaking, devoted to a systematic investigation of *families* of objects in Berkovich's non-Archimedean analytic geometry (see [**Ber90**], [**Ber93**]). More precisely, let us assume that we are given a morphism $Y \to X$ between analytic spaces, and an object D in Y of a certain kind (think of the space Y itself, or a coherent sheaf, or a complex of coherent sheaves...). Every point x of X gives then rise to an object D_x, living on the fiber Y_x, and we thus get in some sense an analytic family of objects parametrized by the space X. The quite vague problem we would like to address is the following: *how do the object D_x and its relevant properties vary?* Of course, such questions have been intensively studied for a long time in *algebraic* geometry, especially by Grothendieck and his school, and our guideline has been to establish analytic avatars of their results every time it was possible. Let us now give a quick overview of our work.

1. First step: flatness in the Berkovich setting

1.1. Motivation. — In scheme theory, the key notion upon which the study of families is based is *flatness*. This is a property of families of *coherent sheaves*, which encodes more or less the intuitive idea of a *reasonable* variation (this is why there is almost always a flatness assumption in the description of moduli problems).

The point is that the study of general families is often reduced (typically, through a suitable stratification of the base scheme) to the case where some of the coherent sheaves involved are flat over the parameter space, which is easier to handle.

But we would like to emphasize that flatness is also a crucial technical tool for many other purposes. Let us mention for example:

- descent theory;
- the first occurrence of flatness in algebraic geometry, in the celebrated paper GAGA by Serre [**Ser56**], where the following plays a major role: if X is a complex algebraic variety and if X^{an} denotes the corresponding analytic space, then for every $x \in X(\mathbf{C})$ the ring $\mathscr{O}_{X^{\mathrm{an}},x}$ is flat over $\mathscr{O}_{X,x}$.

By analogy, our first step toward the understanding of analytic families has been the development of a theory of flatness in Berkovich geometry, which is the core of this book; and similarly to what happens for schemes, we hope that it will have many applications beyond the study of families.

In fact, flatness in non-Archimedean geometry had already been considered, but in the *rigid-analytic* setting, following ideas of Raynaud; see Bosch [**BL93a**] and Lütkebohmert [**BL93b**], as well as the more comprehensive recent study Abbes [**Abb10**]. We now give a more precise discussion of rigid-analytic flatness, before saying some words concerning our definition in the Berkovich framework.

1.2. Flatness in rigid geometry. — The definition of flatness in the rigid setting is as simple as one may hope: if $f : Y \to X$ is a morphism between rigid spaces and if $\mathscr{F}$ is a coherent sheaf on Y, it is rig-flat over X at a point $y \in Y$ if it is flat over X at y *in the sense of the theory of locally ringed spaces*; *i.e.*, the stalk $\mathscr{F}_y$ is a flat $\mathscr{O}_{X,f(y)}$-module (this is the definition by Abbes; the original one by Bosch and Lütkebohmert was slightly different, but both are easily seen to be equivalent using the good properties of the completion of local rings as far as flatness is concerned, *cf.* [**SGA1**], Exposé IV, Cor. 5.8).

Flatness in the above sense behaves well: it is stable under base change and ground field extension. But contrary to what happens in scheme theory, this is in no way obvious, because base change and ground field extension are defined using *completed* tensor products. Roughly speaking, the proofs proceed as follows (see [**BL93a**], [**BL93b**] and [**Abb10**]):

— the study of *rigid* flatness is reduced to that of *formal* flatness through formal avatars of Raynaud-Gruson flattening techniques, which are used to build a flat formal model of any given rig-flat coherent sheaf;

— the study of *formal* flatness is reduced to that of *algebraic* flatness, in a more standard way, upon dividing by various ideals of definition and using flatness criteria in the spirit of [**SGA1**], Exposé IV.

Let us mention that this general strategy (formal flattening and reduction modulo an ideal of definition to replace an analytic problem with an algebraic one) was also used by Raynaud to prove the following fact: if $\varphi : Y \to X$ is a flat morphism between affinoid rigid spaces, $\varphi(Y)$ is a finite union of affinoid domains of X (*cf.* [**BL93b**], Cor. 5.11).

1.3. Flatness in Berkovich geometry. — We fix from now on and for the remaining part of this introduction a complete, non-Archimedean field k. We shall only consider *Berkovich* analytic spaces. Any analytic space X comes with a usual topology, but also with a set-theoretic Grothendieck topology which refines it, the so-called G-*topology* – the corresponding site is denoted by X_{G}; the archetypal example of a G-covering is